Social Security and the Interactions between Aggregate and Idiosyncratic Risk *

Daniel Harenberg[a] and Alexander Ludwig[b]

[a] *ETH Zurich*
[b] *CMR & FiFo, University of Cologne; MEA and Netspar*

This Version: July 15, 2013

Preliminary and incomplete. Comments Welcome!

Abstract

We study the welfare effects of social security in an overlapping generations general equilibrium model with aggregate and idiosyncratic risk. Prior research on social security has only considered either risk in isolation. We show analytically that both risks interact due to the life-cycle structure of the economy. This interaction increases the welfare gains of a marginal introduction of an unfunded social security system. Adding a second interaction by making the variance of idiosyncratic risk countercyclical further increases the welfare gains. In our quantitative experiment, raising the contribution rate from zero to two percent leads to long-run welfare gains of 3.5% of life-time consumption on average, even though the economy experiences substantial crowding out of capital. Approximately one third of these gains can be attributed to the interactions between idiosyncratic and aggregate risk.

JEL classification: C68; E27; E62; G12; H55
Keywords: social security; idiosyncratic risk; aggregate risk; welfare

1   Introduction

Many countries operate large social security systems. One reason is that social security can provide insurance against risks for which there are no private markets. However, these systems also impose costs by distorting prices and decisions. The question arises whether the benefits of social security outweigh the costs.

We address this question in a model which features both aggregate and idiosyncratic risk. We follow the literature and assume that insurance markets for both forms of risk are incomplete. In such a setting social security can increase economic efficiency by partially substituting for missing markets. The analysis is embedded in a general equilibrium framework to account for the costs of crowding out. The difference to the previous literature is that, so far, only models with one kind of risk were examined. One strand of the literature has looked at social security when only aggregate risk is present (e.g. Krueger and Kubler (2006)). There, social security can improve the intergenerational sharing of aggregate risks. The other strand included only idiosyncratic risk (e.g. Imrohoroğlu, Imrohoroğlu, and Joines (1995, 1998)). There, social security is valuable because of intragenerational insurance. However, households face both kinds of risk over their lifetime. To get a more complete picture of how much insurance social security can provide, the different risks need to be included in one model. By doing that, we can assess the contribution of each risk to total insurance. More importantly, we can analyze the role played by interactions between the two types of risk.

The first interaction is an interaction over the life-cycle and accordingly we call it the life-cycle interaction (LCI). To better understand this new effect, consider a standard model in which idiosyncratic wage risk is statistically independent of aggregate risk. Due to the nature of a life-cycle economy, aggregate and idiosyncratic risks directly interact despite their statistical independence. The reason is that when retired, consumption is mainly financed out of private savings. The level of private savings depends on the realizations of idiosyncratic wage risk and aggregate return risk during working life. As a consequence, the variance of private savings contains an interaction term between idiosyncratic and aggregate risk. Because households face these risks for many years before they go into retirement, this interaction term becomes large.

The second interaction operates via the so-called counter-cyclical cross-sectional variance of idiosyncratic productivity shocks (CCV). This means that the variance of idiosyncratic shocks is higher in a downturn than in a boom. The CCV has been documented in the data (Storesletten, Telmer, and Yaron (2004b)), and has been analyzed with respect to asset pricing

(Mankiw (1986), Constantinides and Duffie (1996), Storesletten, Telmer, and Yaron (2007)). We want to understand whether social security can provide insurance against this interaction.

In order to evaluate how much these interactions matter quantitatively, we build a large-scale overlapping generations (OLG) model in the tradition of Auerbach and Kotlikoff (1987), extended by various forms of risk. Aggregate wage risk is introduced through a standard shock to total factor productivity (TFP). Aggregate return risk is introduced through a depreciation shock. The two shocks enable us to calibrate the model in such a way that it produces realistic fluctuations of wages and returns, both of which are central to the welfare implications of social security.

The social security system is a pure pay-as-you-go (PAYG) system. Every period, all the contributions are paid out as a lump-sum to all the retirees. Households can also save privately by investing in a risk-free bond and risky stock. Having this portfolio choice in the quantitative model is important, because social security can be seen as an asset with a low risk and a low return. Therefore, the risk-return structure of the bond and the stock directly affect the value of social security. In order to match a high expected risky return and a low risk-free rate at the same time we need Epstein-Zin-preferences. Finally, households also face survival risk. Therefore, they value social security because it partially substitutes for missing annuity markets.

Our experiment consists of a marginal introduction of social security. We use a two-period model to expose the new life-cycle interaction LCI. We show analytically that social security provides insurance against both LCI and against the countercyclical variance CCV. We also show analytically that the benefit of the insurance against CCV becomes larger when aggregate risk in the economy increases.

When we calibrate the model to the U.S. economy, we find that the introduction of social security leads to a strong welfare gain. This stands in contrast to the previous literature, because social security in our model provides insurance against both idiosyncratic and aggregate risk, as well as their interactions. To be precise, increasing the contribution rate from zero to two percent leads to welfare gains of 3.17% in terms of consumption equivalent variation. This welfare improvement is obtained even though we observe substantial crowding out of capital. About one third of the welfare gains is attributed to the two interactions LCI and CCV.

The welfare gains are not caused by reducing an inefficient overaccumulation of capital. To control for that, we ensure in our calibration that the economy is dynamically efficient.

The idea that social security can insure against aggregate risks goes back to Diamond (1977) and Merton (1983). They show how it can partially

complete financial markets and thereby increase economic efficiency. Building on these insights, Shiller (1999) and Bohn (2001, 2009) show that social security can reduce consumption risk of all generations by pooling labor income and capital income risks across generations if labor income and capital returns are imperfectly correlated.

Gordon and Varian (1988), Ball and Mankiw (2007), Matsen and Thogersen (2004) and Krueger and Kubler (2006) use a two-period partial equilibrium model where households consume only in the second period of life, i.e. during retirement. For our analytical results, we extend this model by adding idiosyncratic risk.

Quantitative papers with aggregate uncertainty and social security are scarce. Krueger and Kubler (2006) is the closest to us.[1] They also look at a marginal introduction of a PAYG system and find that it does not constitute a Pareto-improvement. The concept of a Pareto-improvement requires that they take an ex-interim welfare perspective, whereas we calculate welfare from an ex-ante perspective. Our paper differs in that it adds idiosyncratic risks and analyzes the interactions.

Quantitative papers with idiosyncratic uncertainty and social security, on the other hand, are plenty (e.g. Conesa and Krueger (1999), Imrohoroğlu, Imrohoroğlu, and Joines (1995, 1998), Huggett and Ventura (1999) and Storesletten, Telmer, and Yaron (1999)). On a general level, a conclusion from this literature is that welfare in a stationary economy without social security is higher than in one with a PAYG system. That is, the losses from crowding out dominate the gains from completing insurance markets. The more recent work by Nishiyama and Smetters (2007) and Fehr and Habermann (2008) are examples of papers which focus on modeling institutional features of existing social security systems in detail. Our approach is less policy oriented than theirs and we abstract from such details. Our results show the benefits of a flat pension scheme without additionally optimizing over the exact design of the pension benefit formula.

Huggett and Parra (2010) argue that it is important to look at a simultaneous reform of both the social security system and of the general tax system. They report strong welfare gains from joint reforms of both systems. We instead follow the more standard approach and take the general income tax system as given. Consequently, we calibrate our model to income processes after taxation. Finally, Gomes, Michaelides, and Polkovnichenko (2008) use a very similar model to study how changes in fiscal policy and government

---

[1]Ludwig and Reiter (2010) ask how pension systems should optimally adjust to demographic shocks. Olovsson (2010) claims that pension payments should be very risky because this increases precautionary savings and thereby welfare improving capital formation.

debt affect asset prices and capital accumulation.

The remainder of this paper is structured as follows. We derive our analytical results in section 2. Section 3 develops the quantitative model and section 4 presents the calibration. The main results of our quantitative analysis are presented in section 5, where we make much use of our analytical results. We conclude in section 6. Proofs, computational details, and robustness checks are relegated to separate appendices.

2   A Two-Generations Model

We first develop an analytical model that provides useful insights for our quantitative analysis. We adopt the partial equilibrium framework of Gordon and Varian (1988), Ball and Mankiw (2007), Matsen and Thogersen (2004), Krueger and Kubler (2006) and others who assume that members of each generation consume only in the second period of life. We show that the aforementioned literature—which focuses on aggregate risk only—misses important interaction mechanisms between idiosyncratic and aggregate risk.

As shown in Harenberg, Ludwig, and Maus (2013) such a two period model misses an important aspect of the inter-temporal nature of the savings problem which biases results against social security if wages and returns are positively correlated.[2] To avoid this discussion here—which would in any case lead us on a sidetrack—, we simply shut down the correlation between wages and returns.

2.1   Households

Each period t, a continuum of households is born. Households live for two periods only. A household has preferences over consumption in the second period. In the first period of life, the household experiences an idiosyncratic productivity shock which we denote by η. This shock induces heterogeneity by household type which we denote by i. In addition, we index age by $j =$

---

[2]The intuition is simple. The conventional view is a "hedge view" according to which social security is a better instrument when wages and returns are negatively correlated. The non-conventional view highlighted in Harenberg, Ludwig, and Maus (2013) is a "variance view". A positive correlation between wages and returns increases the variance of total income during the working period when households build up assets. Via taxing wages social security reduce income and thereby also its variance today and shifts this variance to the future by conditioning future social security payments on wage income of future workers. If discounting is sufficiently strong households prefer to hold that risk in the future rather than today. Social security can thereby be welfare improving even when wages and returns are perfectly positively correlated.

$1, 2$. Consequently, all variables at the individual level carry indices i, j, t. The expected utility function of a household born in period t is given by

$$E_t \left[u(c_{i,2,t+1}) \right],$$

where the per period Bernoulli utility function is (weakly) increasing and concave, i.e., $u' > 0, u'' < 0$. Specifically, we assume a CCRA utility function with coefficient of relative risk aversion θ,

$$u(c_{i,2,t+1}) = \frac{c_{i,2,t+1}^{1-\theta} - 1}{1 - \theta}.$$

As households only value second period consumption, consumption and savings decisions are given by

$$s_{i,1,t} = (1 - \tau)\eta_{i,1,t} w_t \tag{1a}$$

$$c_{i,2,t+1} = s_{i,1,t}(1 + r_{t+1}) + b_{t+1} \tag{1b}$$

where $\eta_{i,1,t}$ is the idiosyncratic shock to wages in the first period of life. We assume that $E\eta_{i,1,t} = 1$ for all i, t. b_{t+1} are social security benefits to be specified next and τ is the contribution rate to social security.

2.2 Government

The government organizes a PAYG financed social security system. Pension benefits are lump-sum. Then the social security budget constraint writes as

$$b_t N_{2,t} = \tau w_{t+1} N_{1,t}$$

where $N_{j,t}$ is the population in period t of age j, i.e., $N_{j,t} = \int N_{i,j,t} di$. We ignore population growth, hence

$$b_t = \tau w_t.$$

We can therefore rewrite consumption in the second period as

$$c_{i,2,t+1} = s_{i,1,t}(1 + r_{t+1}) + w_{t+1}\tau.$$

2.3 Welfare

We take an ex-ante Rawlsian perspective and hence specify the social welfare function (SWF) of a cohort born in period t as the expected utility of a generation from the perspective of period $t - 1$:

$$SWF_t \equiv E_{t-1} \left[u(c_{i,2,t+1}) \right].$$

2.4 Stochastic Processes

Wages and interest rates are stochastic. We denote by ζ_t the shock on wages and by $\tilde{\varrho}_t$ the shock on returns. We further assume that wages grow deterministically at rate g. We therefore have:

$$w_t = \bar{w}_t \zeta_t = \bar{w}_{t-1}(1+g)\zeta_t$$
$$R_t = \bar{R}\tilde{\varrho}_t$$

To simplify the analysis we assume that both ζ_t and $\tilde{\varrho}_t$ are not serially correlated. Despite the observed positive serial correlation of wages and asset returns in annual data, this assumption can be justified on the grounds of the long factual periodicity of each period in a two-period OLG model which is about 30 to 40 years. As discussed previously, we also assume that ζ_t and $\tilde{\varrho}_t$ are statistically independent. The idiosyncratic shock $\eta_{i,1,t}$ is not correlated with either of the two aggregate shocks. We relax this assumption once we introduce the CCV mechanism below. All shocks are assumed to have bounded support. We now summarize these assumptions:

Assumption 1. *a) Bounded support: $\zeta_t > 0$, $\tilde{\varrho}_t > 0$ for all t, $\eta_{i,1,t} > 0$ for all i, t.*

b) Means: $E\zeta_t = E\tilde{\varrho}_t = E\eta_{i,1,t} = 1$, for all i, t.

c) Statistical independence of (ζ_{t+1}, ζ_t) and $(\tilde{\varrho}_{t+1}, \tilde{\varrho}_t)$. Therefore: $E(\zeta_{t+1}\zeta_t) = E\zeta_{t+1}E\zeta_t$ for all t and, correspondingly, $E(\tilde{\varrho}_{t+1}\tilde{\varrho}_t) = E\tilde{\varrho}_{t+1}E\tilde{\varrho}_t$ for all t.

d) Statistical independence of $(\zeta_t, \tilde{\varrho}_t)$. Therefore: $E(\zeta_t\tilde{\varrho}_t) = E\zeta_t E\tilde{\varrho}_t$ for all t.

e) Statistical independence of $(\zeta_t, \eta_{i,1,t})$. Therefore: $E(\eta_{i,1,t}\zeta_t) = E\eta_{i,1,t}E\zeta_t$ for all i, t.

f) Statistical independence of $(\tilde{\varrho}_t, \eta_{i,1,t})$. Therefore: $E(\eta_{i,1,t}\varrho_t) = E\eta_{i,1,t}E\varrho_t$ for all i, t.

2.5 Analysis

Life-Cycle Interaction

Given that $c_{i,1,t} = 0$, consumption in the second period can be rewritten as

$$c_{i,2,t+1} = \bar{w}\left(\eta_{i,1,t}\zeta_t\bar{R}\tilde{\varrho}_{t+1} + \tau\left((1+g)\zeta_{t+1} - \eta_{i,1,t}\zeta_t\bar{R}\tilde{\varrho}_{t+1}\right)\right). \qquad (3)$$

We then have:

Proposition 1. *A marginal introduction of social security increases ex-ante expected utility if*

$$(1+g)\frac{E_{t-1}\left[\frac{\zeta_{t+1}}{\tilde{\varrho}_{t+1}^{\theta}}\right]E_{t-1}\left[\frac{1}{\zeta_t^{\theta}}\right]E_{t-1}\left[\frac{1}{\eta_{i,1,t}^{\theta}}\right]}{E_{t-1}\left[\tilde{\varrho}_{t+1}^{1-\theta}\right]E_{t-1}\left[\zeta_t^{1-\theta}\right]E_{t-1}\left[\eta_{i,1,t}^{1-\theta}\right]} > \bar{R}. \tag{4}$$

The RHS of equation (4) reflects the costs of introducing social security represented here by the ex-risk return on savings. We speak of the LHS of equation (4) as the risk-adjusted implicit return of social security which reflects the value (or benefit) of introducing social security. Obviously, the implicit return increases if g increases. This is the standard Aaron condition.

To interpret the risk adjustment, we next assume that all stochastic variables are jointly distributed as log-normal.

Assumption 2. *Joint log-normality:* $\eta_{i,1,t}, \zeta_t, \zeta_{t+1}, \tilde{\varrho}_{t+1}$ *are jointly distributed as log-normal with parameters* $\mu_{\ln \eta}$, $\mu_{\ln \zeta}$, $\mu_{\ln \tilde{\varrho}}$, $\sigma^2_{\ln(\eta)}$, $\sigma^2_{\ln(\zeta)}$, $\sigma^2_{\ln(\tilde{\varrho})}$ *for means and variances, respectively.*

We then have:

Proposition 2. *Under assumption 2, a marginal introduction of social security increases ex-ante expected utility if*

$$(1+g)\cdot(1+TR)^{\theta} > \bar{R}, \tag{5}$$

where

$$TR \equiv var(\eta_{i,1,t}\zeta_t\tilde{\varrho}_{t+1}) = \underbrace{\sigma_{\eta}^2}_{IR} + \underbrace{\sigma_{\zeta}^2 + \sigma_{\tilde{\varrho}}^2 + \sigma_{\zeta}^2\sigma_{\tilde{\varrho}}^2}_{AR} + \underbrace{\sigma_{\eta}^2\left(\sigma_{\zeta}^2 + \sigma_{\tilde{\varrho}}^2 + \sigma_{\zeta}^2\sigma_{\tilde{\varrho}}^2\right)}_{LCI=IR\cdot AR}. \tag{6}$$

To interpret this condition, observe that, according to equation (6), term TR—standing in for "total risk"—consists of three components, reflecting the effect of idiosyncratic risk in term IR, total aggregate risk in term AR and a mechanical interaction between idiosyncratic and aggregate risk in term LCI. To understand the nature of these terms notice that, in absence of social security, savings cum interest in our simple model is given by $s_{i,1,t}R_{t+1} = \bar{w}_t\bar{R}\eta_{i,1,t}\zeta_t\tilde{\varrho}_{t+1}$. Hence, from the ex-ante perspective, the product of three sources of risk are relevant, idiosyncratic wage risk, $\eta_{i,1,t}$, aggregate wage risk, ζ_t, and aggregate return risk, $\tilde{\varrho}_{t+1}$. Term TR is the variance of the product of these stochastic elements. It can be derived by applying the product formula of variances presented in Goodman (1960).

For standard random variables, an interaction term involving products of variances—such as LCI in our context—would be small. However, we

here deal with long horizons so that the single variance terms might be quite large. Illustration 1 in the appendix gives a simple numerical example which uses parameters of our calibrated income processes. Based on this example we conclude that LCI adds about 40 percent times AR. Whatever the exact size of AR is, this interaction is clearly a non-negligible increase in overall income risk.

We next address how the utility consequences of a marginal introduction of social security—by a percentage point increase of $d\tau$—translate into utility. To measure this we compute the consumption equivalent variation (CEV). That is, we express utility gains from introducing social security at rate $d\tau > 0$ as the compensation in a policy regime without social security ($\tau = 0$) in units of a percent increase of consumption g_c. We denote by $g_c(AR, IR)$ the CEV required if both risks, idiosyncratic and aggregate, are present. We decompose this total CEV into various components. We accordingly denote the CEV in a deterministic setting by $g_c(0,0)$, with only aggregate risk by $g_c(AR, 0)$ and with only idiosyncratic risk by $g_c(0, IR)$, respectively. Observe from these definitions that $g_c(AR, 0) = g_c(0,0) + dg_c(AR)$, where $dg_c(AR)$ denotes the additional CEV due to aggregate risk. Correspondingly, we have $g_c(0, IR) = g_c(0,0) + dg_c(IR)$. With these definitions, it is also straightforward to define the additional effects, in terms of CEV, of the interaction between idiosyncratic and aggregate risks. It is given as the residual, namely, $dg_c(LCI) = g_c(AR, IR) - (g(0,0) + dg_c(AR) + dg_c(IR))$. In the appendix, we show that $g_c(AR, IR)$ can be expressed—in a logarithmic approximation—as

$$g_c = \left(\frac{1+g}{\bar{R}}(1+V)^\theta - 1 \right) d\tau \tag{7}$$

Taking a first-order Taylor series expansion of the above around $V = 0$ gives

$$g_c(AR, IR) \approx \left(\underbrace{\frac{1+g}{\bar{R}} - 1}_{g_c(0,0)/d\tau} + \underbrace{\theta\frac{1+g}{\bar{R}}AR}_{dg_c(AR)/d\tau} + \underbrace{\theta\frac{1+g}{\bar{R}}IR}_{dg_c(IR)/d\tau} + \underbrace{\theta\frac{1+g}{\bar{R}}LCI}_{dg_c(LCI)/d\tau} \right) d\tau \tag{8}$$

These equations have a straightforward interpretation. First, utility losses in a dynamically efficient economy—where $\bar{R} > 1 + g$—are approximately linear in the size of the social security system, $d\tau$. Additional gains due to insurance against risk—the risk components being AR, IR and LCI, respectively—increase in the size of risk whereby this increase is exponentially in risk aversion, cf. equation (7). Finally, the proportional increase of risk via the interaction translates—in a first-order approximation given by

equation (8)—into corresponding utility consequences as measured by CEV because $dg_c(LCI) = IR \cdot dg_c(AR)$.

Modification: Counter-Cyclical Conditional Variance

We now return to condition (4) and modify assumption 2 slightly in order to reflect the CCV mechanism. Observe that CCV, by definition, does away with assumption 1e.

Assumption 3. $\zeta_t \in [\zeta_l, \zeta_h]$ *for all t where $\zeta_h > \zeta_l > 0$. We let $\zeta_h = 1 + \Delta_\zeta$ and $\zeta_l = 1 - \Delta_\zeta$ where $\Delta_\zeta < 1$. Notice that $\frac{1}{2}(\zeta_l + \zeta_h) = 1$. $\eta_{i,1,t}$ is distributed as log-normal whereby*

$$\eta_{i,1,t} = \begin{cases} \eta_{i,1,l} & \text{for } \zeta_t = \zeta_l \\ \eta_{i,1,h} & \text{for } \zeta_t = \zeta_h. \end{cases}$$

and $E \ln \eta_{i,1,l} = E \ln \eta_{i,j,h} = E \ln \eta_{i,j,t} = E \ln \eta$ and

$$\sigma^2_{\ln \eta} = \begin{cases} \sigma^2_{\ln \eta h} = \sigma^2_{\ln \eta} + \Delta & \text{for } \zeta_t = \zeta_l \\ \sigma^2_{\ln \eta l} = \sigma^2_{\ln \eta} - \Delta & \text{for } \zeta_t = \zeta_h. \end{cases}$$

For simplicity, we focus only at the log-uility case, hence $\theta = 1$. The RHS of equation (4) then rewrites as

$$(1+g)E_{t-1}\left[\frac{\zeta_{t+1}}{\tilde{\varrho}_{t+1}}\right] E_{t-1}\left[\frac{1}{\zeta_t}\right] E_{t-1}\left[\frac{1}{\eta_{i,1,t}}\right]$$

Under assumption 3, the expression rewrites as

$$(1+g)E_{t-1}\left[\frac{\zeta_{t+1}}{\tilde{\varrho}_{t+1}}\right] \frac{1}{2}\left(\frac{1}{\zeta_l}E_{t-1}\left[\frac{1}{\eta_{i,1,l}}\right] + \frac{1}{\zeta_h}E_{t-1}\left[\frac{1}{\eta_{i,1,h}}\right]\right) \qquad (9)$$

and, without CCV, the corresponding expression is

$$(1+g)E_{t-1}\left[\frac{\zeta_{t+1}}{\tilde{\varrho}_{t+1}}\right] \frac{1}{2}\left(\frac{1}{\zeta_l} + \frac{1}{\zeta_h}\right) E_{t-1}\left[\frac{1}{\eta_{i,j,t}}\right]. \qquad (10)$$

We can then show the following:

Proposition 3. *a) the LHS of eq. (9) is larger than the LHS of eq. (10).*

b) the difference between the LHS of eq. (9) and the LHS of eq. (10) increases in the variance of aggregate shocks.

We can therefore conclude that, on top of the previously illustrated mechanical interaction between idiosyncratic and aggregate risk, the direct interaction via the CCV mechanism will further increase the beneficial effects of social security. Importantly, finding 3b establishes that the effect of CCV is larger when the variance of aggregate risk is higher.

2.6 Extension: Analysis in General Equilibrium

Our simple setup only provides a partial characterization of the total welfare effects of social security. As we do not consider consumption in the first period of life, it misses the effects of taxation on reallocation of consumption and savings. Furthermore, we do not consider any feedback in general equilibrium. In our companion paper, Harenberg and Ludwig (2013), we incorporate both channels in a standard Diamond (1965) model with risk.[3] Hence, relative to the simple model presented here, consumption and savings decisions take place in the first period and wages and returns are determined in general equilibrium. There, we conclude with two additional findings that directly follow from the intuition provided above. First, by providing insurance social security reduces precautionary savings so that first period consumption increases and the consumption growth rate goes down. Interaction of risks increases the welfare benefits arising from such a reallocation of resources. Second—as a mirror image of this effect—interactions of risk drive up the welfare losses induced by crowding out of capital. Our quantitative model to which we turn next accounts for all these channels.

3 The Quantitative Model

[TBC]

Our quantitative model extends our simple model along several dimensions. First, we take a general equilibrium perspective. Second, rather than considering a stylized setup with two generations we take a periodicity of one calendar year and consider J overlapping generations. Consumption and savings decisions take place every period. Third, we introduce one period ahead risk-free bonds. The primary reason for this extension is to impose discipline on calibration. Having a bond in the model means that our model entails predictions about general equilibrium asset prices. Any model on the welfare effects of social security should have realistic asset pricing implications. By providing a bond, we give households an additional asset to self-insure against idiosyncratic and aggregate risk. Ceteris paribus, this reduces the beneficial effects of social security. However, the presence of the bond also reduces the effect of decreasing savings on the crowding out of productive capital because part of the reduced savings is absorbed by the bond market.

---

[3]These extensions involve extensive algebra. For reasons of space we relegate this analysis to the companion paper.

3.1 Risk and Time

Time is discrete and runs from $t = 0, \ldots, \infty$. Risk is represented by an event tree. The economy starts with some fixed event z_0, and each node of the tree is a history of exogenous shocks $z^t = (z_0, z_1, ..., z_t)$. The shocks are assumed to follow a Markov chain with finite support Z and strictly positive transition matrix $\pi^z(z' \mid z)$. Let Π^z denote the invariant distribution associated with π^z. In our notation, we will make all aggregate and idiosyncratic shocks contingent on z_t. For notational convenience, we will suppress the dependency of all other variables on z^t but history dependence of all choice variables is understood.

3.2 Demographics

In each period t, the economy is populated by J overlapping generations of agents indexed by $j = 1, \ldots, J$, with a continuum of agents in each generation. Population grows at the exogenous rate of n. Households face an idiosyncratic (conditional) probability to survive from age j to age $j + 1$ which we denote by ς_{j+1}, hence $\varsigma_1 = 1$ and $\varsigma_{J+1} = 0$. Consequently, given an initial population distribution $\{N_{0,j}\}_{j=1}^J$ which is consistent with constant population growth for all periods $t = 0, 1, \ldots$ and normalized such that $N_0 = \sum_{j=1}^J N_{t,j} = 1$, the exogenous law of motion of population in our model is given by

$$N_{t+1,1} = (1 + n)N_{t,1}$$
$$N_{t+1,j+1} = \varsigma_{j+1} \cdot N_{t,j} \quad \text{for } j = 1, \ldots, J.$$

Households retire at the fixed age j_r. Labor supply is exogenous in our model and during the working period $j = 1, \ldots, j_r - 1$ each household supplies one unit of labor. Observe that constant population growth implies that population shares, e.g., the working age to population ratio, are constant.

3.3 Firms

Production of the final good takes place with a standard Cobb-Douglas production function with total output at time t given by

$$Y_t = F(\zeta(z_t), K_t, L_t) = \zeta(z_t)K_t^\alpha(\Upsilon_t L_t)^{1-\alpha} \tag{11}$$

where K_t is the aggregate stock of physical capital, L_t is labor, $\zeta(z_t)$ is a stochastic shock to productivity and Υ_t is a deterministic technology level growing at the exogenous rate g.

The economy is closed. The consumption good can either be consumed in the period when it is produced or can be used as an input into a production technology producing capital. We ignore capital adjustment costs. Accordingly, the production technology for capital is

$$
\begin{aligned}
K_{t+1} &= I_t + K_t(1 - \delta(z_t)) \\
&= Y_t - C_t + K_t(1 - \delta(z_t)) \tag{12}
\end{aligned}
$$

where $\delta(z_t)$ is the stochastic depreciation rate of physical capital.

Firms maximize profits and operate in perfectly competitive markets. Accordingly, the rate of return to capital and the wage rate are given by

$$
w_t = (1 - \alpha)\Upsilon_t\zeta(z_t)k_t^\alpha \tag{13a}
$$

$$
r_t = \alpha\zeta(z_t)k_t^{\alpha-1} - \delta(z_t) \tag{13b}
$$

where $k_t = \frac{K_t}{\Upsilon_t L_t}$ is the capital stock per unit of efficient labor which we refer to as "capital intensity".

3.4 Endowments

Agents are endowed with one unit of labor which is supplied inelastically for ages $j = 1, \ldots, j_r - 1$. After retirement, labor supply is zero. Households have access to two savings storage technologies. Either they save in the risky technology at rate of return r_t or in a one-period risk-free bond at return r_t^f which is in zero net supply. Households are subject to idiosyncratic shocks to their labor productivity. This shock induces heterogeneity by household type which we denote by i. We denote total assets by $A_{i,j,t}$, and the share invested in the risky asset by $\kappa_{i,j,t}$.

Additional elements of the dynamic budget constraint are income, $Y_{i,j,t}$, to be specified below and consumption, $C_{i,j,t}$. The dynamic budget constraint of a household at age j then reads as

$$
A_{i,j+1,t+1} = A_{i,j,t}(1 + r_t^f + \kappa_{i,j-1,t-1}(r_t - r_t^f)) + Y_{i,j,t} - C_{i,j,t} \tag{14}
$$

where $\kappa_{i,j-1,t-1} \in [-\underline{\kappa}, \overline{\kappa}]$, for all i, j, t. This restricts the leverage in stocks in our model.[4]

---

[4]In a model without a constraint of the form $\kappa_{i,j-1,t-1} \in [-\underline{\kappa}, \overline{\kappa}]$ we have a singularity at $X_{i,j,t} - C_{i,j,t} = 0$ so that, for $X_{i,j,t} - C_{i,j,t} \to +0$, $\kappa_{i,j,t} \to +\infty$ and for $X_{i,j,t} - C_{i,j,t} \to -0$, $\kappa_{i,j,t} \to -\infty$. The presence of the singularity has consequences for aggregation because the set for κ will not be compact. We set the constraint in order to rule out this technicality, but we set the bounds so high that the constraint will rarely be binding in equilibrium.

Income is given by

$$Y_{i,j,t} = \begin{cases} (1-\tau)\epsilon_j w_t \eta_{i,j,t} & \text{for } j < jr \\ B_{i,j,t} & \text{for } j \geq jr \end{cases} \tag{15}$$

where ϵ_j is age-specific productivity and $\eta_{i,j,t}$ is an idiosyncratic stochastic component.

We assume that $\eta_{i,j,t}$ follows a time and age-independent Markov chain whereby the states of the Markov chain are contingent on aggregate states z. Accordingly, let the states be denoted by $\mathcal{E}_z = \{\eta_{z1}, \ldots, \eta_{zM}\}$ and the transition matrices be $\pi^\eta(\eta' \mid \eta) > 0$. Let Π^η denote the invariant distribution associated with π^η.

As for pension income, we assume that pension payments are lump-sum, hence

$$B_{i,j,t} = b_t \Upsilon_t \tag{16}$$

where b_t is some normalized pension benefit level which only depends on t. Accordingly, the pension system fully redistributes across household types. This is an approximation to the U.S. pension system.[5]

3.5 Preferences

We take Epstein-Zin preferences. Let θ be the coefficient of relative risk-aversion and φ denote the inter-temporal elasticity of substitution. Then

$$U_{i,j,t} = \left[C_{i,j,t}^{\frac{1-\theta}{\gamma}} + \beta \left(\varsigma_{j+1} \mathbb{E}_{i,j,t} \left[U_{i,j+1,t+1}^{1-\theta} \right] \right)^{\frac{1}{\gamma}} \right]^{\frac{\gamma}{1-\theta}} \tag{17}$$

where $\gamma = \frac{1-\theta}{1-\frac{1}{\varphi}}$, and $\beta > 0$ is the standard discount factor. For $\theta = \frac{1}{\varphi}$ we have $\gamma = 1$ and are back to CRRA preferences. $\mathbb{E}_{i,j,t}$ is the expectations operator and expectations, conditional on information for household i, j, t, are taken with respect to idiosyncratic wage shocks and aggregate productivity and depreciation shocks. We assume that $U_J = C_J$.

3.6 The Government

The government organizes a PAYG financed social security system. We take the position that social security payments are not subject to political risk.

---

[5]The U.S. pension system links contributions to AIME, the average indexed monthly earnings and has an additional distributional component by the so-called bend point formula. From an ex-ante perspective, given this distributional component and provided that income shocks are non-permanent, an approximation with lump-sum pension benefits is a good first-order approximation.

We assume that the budget of the social security system is balanced in all periods. We describe various social security scenarios below. We further assume that the government collects all accidental bequests and uses them up for government consumption which is otherwise neutral.

3.7 Equilibrium

To define equilibrium we adopt a de-trended version of the household model. We therefore first describe transformations of the household problem and then proceed with the equilibrium definition.

Transformations

Following Deaton (1991), define cash-on-hand by $X_{i,j,t} = A_{i,j,t}(1 + r_t^f + \kappa_{i,j-1,t-1}(r_t - r_t^f)) + \mathrm{Y}_{i,j,t}$. The dynamic budget constraint (14) then rewrites as

$$X_{i,j+1,t+1} = (X_{i,j,t} - C_{i,j,t})(1 + r_{t+1}^f + \kappa_{i,j,t}(r_{t+1} - r_{t+1}^f)) + \mathrm{Y}_{i,j+1,t+1} \quad (18)$$

We next transform the problem to de-trend the model and work with stationary variables throughout. That is, we de-trend with the deterministic trend component induced by technological progress. Along this line, define by $x_{i,j,t} = \frac{X_{i,j,t}}{\Upsilon_t}$ transformed cash-on-hand and all other variables accordingly. Using $\omega_t = \frac{w_t}{\Upsilon_t}$ to denote wages per efficiency unit we have

$$\mathrm{y}_{i,j,t} = \begin{cases} (1 - \tau)\epsilon_j \omega_t \eta_{i,j,t} & \text{for } j < jr \\ b_t & \text{for } j \geq jr. \end{cases}$$

Now divide the dynamic budget constraint (18) by Υ_t and rewrite to get

$$x_{i,j+1,t+1} = (x_{i,j,t} - c_{i,j,t})\tilde{R}_{i,j+1,t+1} + \mathrm{y}_{i,j+1,t+1}. \quad (19)$$

where $\tilde{R}_{i,j+1,t+1} = \frac{(1 + r_{t+1}^f + \kappa_{i,j,t}(r_{t+1} - r_{t+1}^f))}{1+g}$.

Transform the per period utility function accordingly and take an additional monotone transformation to get

$$u_{i,j,t} = \left[c_{i,j,t}^{\frac{1-\theta}{\gamma}} + \tilde{\beta}_{j+1} \left(\mathbb{E}_{i,j,t} \left[(u_{i,j+1,t+1})^{1-\theta} \right] \right)^{\frac{1}{\gamma}} \right]^{\frac{\gamma}{1-\theta}} \quad (20)$$

where $\tilde{\beta}_{j+1} = \beta \varsigma_{j+1}^{\frac{1}{\gamma}} (1 + g)^{\frac{1-\theta}{\gamma}}$.

Definition of Equilibrium

Individual households, at the beginning of period t are indexed by their age j, their idiosyncratic productivity state η, their cash on hand holdings x, and a measure $\Phi(j, x, \eta)$ which describes the beginning of period wealth distribution in the economy, i.e., the share of agents at time t with characteristics (j, x, η). We normalize such that $\int d\Phi = 1$. Existence of aggregate shocks implies that Φ evolves stochastically over time. We use H to denote the law of motion of Φ which is given by

$$\Phi' = H(\Phi, z, z') \tag{21}$$

Notice that z' is a determinant of Φ' because it determines $\tilde{R}_{i,j+1,t+1}$ and therefore the distribution over $x\prime$.

The de-trended version of the household problem writes as

$$u(j, x, \eta; z, \Phi) = \max_{c, \kappa, x'} \left\{ \left[c^{\frac{1-\theta}{\gamma}} + \tilde{\beta} \left(\mathbb{E} \left[(u(j+1, x', \eta'; z', \Phi'))^{1-\theta} \right] \right)^{\frac{1}{\gamma}} \right]^{\frac{\gamma}{1-\theta}} \right\}$$

$$\text{s.t. } x' = (x - c)\tilde{R}' + \text{y}'$$

$$\tilde{R}' = \frac{(1 + r^{f'} + \kappa(r' - r^{f'}))}{1 + g}$$

$$\Phi' = H(\Phi, z, z').$$

We therefore have the following definition of the recursive equilibrium of our economy:[6]

Definition 1. *A recursive competitive equilibrium is a value function u, policy functions for the household, $x'(\cdot)$, $a'(\cdot)$, $c(\cdot)$, $\kappa(\cdot)$, policy functions for the firm, $K(\cdot)$, $L(\cdot)$, pricing functions $r(\cdot)$, $q(\cdot)$, $w(\cdot)$, policies, τ, b, aggregate measures $\Phi(\cdot)$ and an aggregate law of motion, H_t such that*

1. *$u(\cdot), x'(\cdot), a'(\cdot), c(\cdot), \kappa(\cdot)$ are measureable, $u(\cdot)$ satisfies the household's recursive problem and $x'(\cdot), a'(\cdot), c(\cdot), \kappa(\cdot)$ are the associated policy functions, given r, q, ω, τ and b.*

2. *K, L satisfy, given $r(\Phi, z)$ and $w(\Phi, z)$,*

$$\omega(\Phi, z) = (1 - \alpha)\zeta(z)k(\Phi, z)^\alpha \tag{23a}$$

$$r(\Phi, z) = \alpha\zeta(z)k(\Phi, z)^{\alpha-1} - \delta(z). \tag{23b}$$

---

[6]We use the integration operator $\int$ as a short-cut notation for all sums and integrals involved but discreteness of the characteristics (j, z) is understood. When integrating out with respect to all characteristics of the distribution, we simply write $d\Phi$, hence $\int \cdot d\Phi = \int \cdot \Phi(dj \times dx \times d\eta)$. When we integrate only with respect to a subset of characteristics, we make this explicit by, e.g., writing $\int \cdot \Phi(j, dx \times d\eta)$.

where $k(\Phi, z) = \frac{K(\Phi, z)}{\Upsilon L}$ is the capital stock per efficiency unit (or "capital intensity") and $\Upsilon = (1 + g)\Upsilon_{-1}$ is the technology level in period t.

3. neutral government consumption financed by bequests is given by

$$gc' = \frac{\int (1 - \varsigma_{j+1})a'(j, x, \eta; z, \Phi) R'(\kappa(\cdot))d\Phi}{\ell(1 + n)(1 + g)} \qquad (24)$$

where

$$R'(\kappa(\cdot)) = (1 + r^{f'} + \kappa(j, x, \eta; z, \Phi)(r' - r^{f'})).$$

4. the pension system budget constraint holds, i.e.

$$\tau(\Phi, z)\omega(\Phi, z) = b(\Phi, z)p \qquad (25)$$

where p is the economic dependency ratio which is stationary in our model.[7]

5. For all Φ and all z

$$k(H(\Phi, z, z'), z')(1 + g)(1 + n) = \frac{1}{\ell}\int \kappa(j, x, \eta; z, \Phi)a'(j, x, \eta; z, \Phi)d\Phi \qquad (26a)$$

$$0 = \int (1 - \kappa(j, x, \eta; z, \Phi))a'(j, x, \eta; z, \Phi)d\Phi \qquad (26b)$$

$$i(\Phi, z) = f(k(\Phi, z)) - gc - \frac{1}{\ell}\int c(j, x, \eta; z, \Phi)d\Phi \qquad (26c)$$

$$k(H(\Phi, z, z'), z')(1 + g)(1 + n) = k(\Phi, z)(1 - \delta(z)) + i(\Phi, z) \qquad (26d)$$

where ℓ is the working age to population ratio[8], equation (26b) is the bond market clearing condition and the bond price q is determined such that it clears the bond market in each period t and $i(\cdot) = \frac{I(\cdot)}{\Upsilon L}$ is investment per efficiency unit.

6. The aggregate law of motion H is generated by the exogenous population dynamics, the exogenous stochastic processes and the endogenous asset accumulation decisions as captured by the policy functions x'.

Definition 2. A stationary recursive competitive equilibrium is as described above but with time constant individual policy functions $x'(\cdot)$, $a'(\cdot)$, $c(\cdot)$, $\kappa(\cdot)$ and a time constant aggregate law of motion H.

---

[7]It is given by $p = \frac{\sum_{j=j_r}^{J}(1+n)^{J-j}\prod_{i=1}^{j-1}\varsigma_i}{\sum_{j=1}^{j_r-1}(1+n)^{J-j}\epsilon_j\prod_{i=1}^{j-1}\varsigma_i}$.

[8]It is given by $\ell = \frac{\sum_{j=1}^{j_r-1}(1+n)^{J-j}\epsilon_j\prod_{i=1}^{j-1}\varsigma_i}{\sum_{j=1}^{J}(1+n)^{J-j}\prod_{i=1}^{j-1}\varsigma_i}$.

3.8 Welfare Criteria

We compare two long-run stationary equilibria and do not take into account transitional dynamics. Our welfare concept is the consumption-equivalent variation for a newborn before any shocks are realized. It is an ex-ante perspective where the agent does not know the aggregate state nor the level of capital that he will be born into. A positive number then states the amount an agent would be willing to give up in order to be born into the second long-run equilibrium (i.e. into an economy with some social security).

Note that this comparison between long-run equilibria provides a lower bound on the expected welfare gains for newborns along the transition, because they are spared some of the negative effects of crowding out, and because they get to save less and consume more as the level of capital moves toward its new, lower level.

3.9 Thought Experiment

In our initial equilibrium a social security system does not exist. In the second equilibrium, the economy features a social security system with a contribution rate of 2 percent. One can think of this as the introduction of a 'marginal' social security system as described in Krueger and Kubler (2006). We use their proposition 1 to ensure that the initial economy is dynamically efficient so as to rule out any welfare gains that would come from curing dynamic inefficiency.

We then use the exact same economy to conduct partial equilibrium (PE) experiments that enable us to disentangle the welfare gains due to insurance from the welfare losses due to crowding out and its associated price changes. In this partial equilibrium, we feed in the sequence of shocks and prices $\{z_t, r_t, r_t^f, w_t\}_{t=1}^T$ obtained from the associated general equilibrium (GE). It is like a small open economy, where aggregate prices are determined by the world and fluctuate over time and are not influenced by domestic policy changes. If we do not change any other parameter, then the results are naturally exactly the same as in the associated GE. To isolate the total insurance effects, we let agents optimize under the new policy, i.e. $\tau = 0.2$, but with the 'old' approximate laws of motion that still hold for the evolution of aggregate prices. Then we simulate by feeding in the old sequence of shocks and prices, but with the new policy functions and the new social security system. In a very similar fashion, we isolate the insurance against aggregate risk, idiosyncratic risk, CCV, and survival risk.

In order to also isolate the interaction effect LCI, we proceed by relating back to equation (8) of our 2-generations model. Recall that $g_c(AR, IR)$—the

total welfare gains with full aggregate and idiosyncratic risk at work, ignoring CCV and survival risk—can be decomposed as $g_c(AR, IR) = g_c(0, 0) + dg_c(AR) + dg_c(IR) + dg_c(LCI)$ where $g_c(0, 0)$ is the welfare gain—expressed in terms of CEV—in an economy with zero aggregate risk and zero idiosyncratic risk and $dg(X)$ is the additional gain attributed to component X, our objects of interest. Also recall that welfare gains in an economy with zero aggregate risk and full idiosyncratic risk, $g_c(0, IR)$, can be written as $g_c(0, IR) = g_c(0, 0) + dg_c(IR)$. Correspondingly we have for an economy with only aggregate risk that $g_c(AR, 0) = g_c(0, 0) + dg_c(AR)$. With objects $g_c(0, 0)$, $g_c(0, IR)$, $g_c(AR, 0)$ and $g_c(AR, IR)$ at hand we can determine $dg(X)$ for $X = AR, IR, LCI$ with the following system of equations

$$
\begin{align}
g_c(0, IR) \quad &= g_c(0, 0) \quad\quad\quad\quad\quad + dg_c(IR) \tag{27a} \\
g_c(AR, 0) \quad &= g_c(0, 0) + dg_c(AR) \tag{27b} \\
g_c(AR, IR) \quad &= g_c(0, 0) + dg_c(AR) \quad + dg_c(IR) + dg_c(LCI) \tag{27c}
\end{align}
$$

This requires solving models without idiosyncratic risk—to determine $g_c(AR, 0)$—and without aggregate risk—to determine $g_c(0, 0)$ and $g_c(0, IR)$. As the latter requires solution of models without aggregate risk, there will no longer be two differential assets with respective returns and no portfolio choice. In these economies, we set the rate of return to 4.2% which corresponds to the empirical estimate of Siegel (2002).

3.10 Computational Details

Following Gomes and Michaelides (2008) and Storesletten, Telmer, and Yaron (2007) we compute an approximate equilibrium of our model by applying the Krusell and Smith (1998) method. We approximate the solution by considering forecast functions of the average capital stock in the economy and the ex-ante equity premium. In the general equilibrium version of our model, we loop on the postulated laws of motion until convergence. We do so by simulating the economy for $T = 5000$ periods and discard the first 500 initialization periods. In each period, we compute the market clearing bond price. The goodness of fit of the approximate laws of motion is $R^2 = 0.99$.

We compute solution to the household model by adopting Carroll's endogenous grid method, which reduces computational time strongly. Written in Fortran 2003, the model takes about one hour to converge to a solution, given a decent initial guess for the laws of motion. A more detailed description of our computational methods can be found in appendix B.

4   Calibration

4.1   Overview

Part of our parameters are exogenously calibrated either by reference to other studies or directly from the data. We refer to these parameters as first stage parameters. A second set of parameters is calibrated by informally matching simulated moments to respective moments in the data. Accordingly, we refer to those parameters as second stage parameters. Table 1 summarizes the calibration.

4.2   Production Sector

We set the value of the capital share parameter, a first stage parameter, to $\alpha = 0.32$. This is directly estimated from NIPA data (1960-2005) on total compensation as a fraction of (adjusted) GDP. Our estimated value is in the range of values considered as reasonable in the literature. It is close to the preferred value of 0.3 as used by Krueger and Kubler (2006). To estimate α, we take data on total compensation of employees (NIPA Table 1.12) and deflate it with the GDP inflator (NIPA Table 1.1.4). In the numerator, we adjust GDP (NIPA Table 1.1.5), again deflated by the GDP deflator, by nonfarm proprietors' income and other factors that should not be directly related to wage. Without these adjustments, our estimate of α would be considerably higher, i.e., at $\alpha = 0.43$.

To determine the mean depreciation rate of capital, a first stage parameter in our model, we proceed as follows. We first estimate the capital output ratio in the economy. To measure capital, we take the stock of fixed assets (NIPA Table 1.1), appropriately deflated. We relate this to total GDP. This gives an estimate of the capital output ratio of $K/Y = 2.65$, in line with the estimates by, e.g., Fernández-Villaverde and Krueger (2011), or of the ratio of output to capital of 0.38. This implies an average marginal product of capital $E[mpk] = \alpha E[Y/K] = 0.12$. Given this estimate for the marginal product of capital and our estimate for the average risky return on capital of 0.079 based on data since 1950 provided by Rob Shiller, we set $E[\delta] = E[mpk] - E[r] = 0.042$.[9]

Our estimate of the deterministic trend growth rate, also a first stage parameter, is $g = 0.018$ which is in line with other studies. We determine it by estimating the Solow residual from the production function, given our estimate of α, our measure for capital, and a measure of labor supply determined

---

[9]The data was downloaded from Rob Shillers webpage and is available under the address http://www.econ.yale.edu/∼shiller/data.htm.

Table 1: Calibration: Summary

Parameter	Value	Target (source)	Stage
Preferences			
Discount factor, β	0.97	Capital output ratio, 2.65 (NIPA)	2
Coefficient of relative risk aversion, θ	8	Average equity premium, 0.056 (Shiller)	2
Elasticity of intertemporal substitution, φ	1.5	Consumption Profile	2
Technology			
Capital share, α	0.32	Wage share (NIPA)	1
Leverage, b	0.66	Croce (2010)	1
Technology growth, g	0.018	TFP growth (NIPA)	1
Mean depreciation rate of capital, δ_0	0.0418	Risky return, 0.079 (Shiller)	1
Std. of depreciation $\bar{\delta}$	0.11	Std. of risky return, 0.168 (Shiller)	2
Aggregate productivity states, $1 \pm \bar{\zeta}$	{1.029, 0.971}	Std. of TFP, 0.029 (NIPA)	1
Transition probabilities of productivity, π^ζ	0.941	Autocorrelation of TFP, 0.88 (NIPA)	1
Conditional prob. of depreciation shocks, π^δ	0.86	Corr.(TFP, returns), 0.36 (NIPA, Shiller)	2
Idiosyncratic Productivity			
Age productivity, $\{\epsilon_j\}$	-	Earnings profiles (PSID)	1
CCV $\sigma_{\nu(z)}$	{0.21, 0.13}	Storesletten, et al. (2007)	1
Autocorrelation ρ	0.952	Storesletten, et al. (2007)	1
Demographics: Exogenous parameters			
Biological age at $j = 1$	21		1
Model age at retirement, j_r	45		1
Model age maximum, J	70		1
Survival rates, $\{s_j\}$	-	Population data (HMD)	1
Population growth, n	0.011	U.S. Social Sec. Admin. (SSA)	1

by multiplying all full- and part-time employees in domestic employment (NIPA Table 6.4A) with an index for aggregate hours (NIPA Table 6.4A).[10] We then fit a linear trend specification to the Solow residual. Acknowledging the labor augmenting technological progress specification chosen, this gives the aforementioned point estimate.

4.3   Aggregate States and Shocks

We assume that aggregate risk is driven by a four state Markov chain with support $Z = \{z_1, \ldots, z_4\}$ and transition matrix $\pi = (\pi_{ij})$. Each aggregate state maps into a combination of low or high technology shocks and low or high physical capital depreciation. We let

$$\zeta(z) = \begin{cases} 1 - \bar{\zeta} \text{ for } z \in z_1, z_2 \\ 1 + \bar{\zeta} \text{ for } z \in z_3, z_4 \end{cases} \quad \text{and } \delta(z) = \begin{cases} \delta_0 + \bar{\delta} \text{ for } z \in z_1, z_3 \\ \delta_0 - \bar{\delta} \text{ for } z \in z_2, z_4. \end{cases} \quad (28)$$

With this setup, z_1 corresponds to a low wage and a low return, while z_4 corresponds to a high wage and a high return.

To calibrate the entries of the transition matrix, denote by $\pi^\zeta = \pi(\zeta' = 1 - \bar{\zeta} \mid \zeta = 1 - \bar{\zeta})$ the transition probability of remaining in the low technology state. Assuming that the transition of technology shocks is symmetric, we then also that $\pi(\zeta' = 1 + \bar{\zeta} \mid \zeta = 1 + \bar{\zeta}) = \pi^\zeta$ and, accordingly $1 - \pi^\zeta = \pi(\zeta' = 1 - \bar{\zeta} \mid \zeta = 1 + \bar{\zeta}) = \pi(\zeta' = 1 + \bar{\zeta} \mid \zeta = 1 - \bar{\zeta})$.

To govern the correlation between technology and depreciation shocks, let the probability of being in the high (low) depreciation state conditional on being in the low (high) technology state, assuming symmetry, be $\pi^\delta = \pi(\delta' = \delta_0 + \bar{\delta} \mid \zeta' = 1 - \bar{\zeta}) = \pi(\delta' = \delta_0 - \bar{\delta} \mid \zeta' = 1 + \bar{\zeta})$. We then have that the transition matrix of aggregate states follows from the corresponding assignment of states in (28) as

$$\pi^z = \begin{bmatrix} \pi^\zeta \cdot \pi^\delta & \pi^\zeta \cdot (1 - \pi^\delta) & (1 - \pi^\zeta) \cdot (1 - \pi^\delta) & (1 - \pi^\zeta) \cdot \pi^\delta \\ \pi^\zeta \cdot \pi^\delta & \pi^\zeta \cdot (1 - \pi^\delta) & (1 - \pi^\zeta) \cdot (1 - \pi^\delta) & (1 - \pi^\zeta) \cdot \pi^\delta \\ (1 - \pi^\zeta) \cdot \pi^\delta & (1 - \pi^\zeta) \cdot (1 - \pi^\delta) & \pi^\zeta \cdot (1 - \pi^\delta) & \pi^\zeta \cdot \pi^\delta \\ (1 - \pi^\zeta) \cdot \pi^\delta & (1 - \pi^\zeta) \cdot (1 - \pi^\delta) & \pi^\zeta \cdot (1 - \pi^\delta) & \pi^\zeta \cdot \pi^\delta \end{bmatrix}$$

In sum, the Markov chain process of aggregate shocks is characterized by four parameters, $(\bar{\zeta}, \bar{\delta}, \pi^\zeta, \pi^\delta)$. All of these parameters are second stage parameters which we calibrate jointly to match the following targets: (i) an average variance of the cyclical component of TFP, again estimated from

---

[10]Notice that we thereby ignore age-specific productivity which should augment our measure of employment.

NIPA data, (ii) the average fluctuation of the risky return which features a standard deviation in the data of 0.16, (iii) the autocorrelation of the cyclical component of TFP in the data and (iv) the estimated correlation of the cyclical component of TFP with risky returns.

As to the latter targets, notice that linear detrending, as assumed, e.g., by Krueger and Kubler (2006), results in a negative correlation of TFP and asset returns as well as wages and asset returns. Table 2 below gives the numbers in column NC, standing in for negative correlation. We regard this finding as unrealistic.[11]

Table 2: Calibration: Estimates of aggregate risk

	NC	PC
Corr. (TFP, returns), $cor(\zeta_t, r_t)$	-0.08 (0.57)	0.50 (0.00)
Corr. (wages, returns), $cor(w_t, r_t)$	-0.33 (0.016)	0.306 (0.025)

Notes: NC: Negative correlation between TFP shocks and returns (linear trend estimation), PC: Positive correlation between TFP shocks and returns (first differences estimation). *p*-values are reported in brackets.

Assuming, on the contrary, a unit root process for (the log of) TFP and detrending by first differences yields a highly significant positive correlation, cf. column PC in table 2. This finding coincides with our economic intuition as we would expect these variables to co-move over the cycle. For sake of consistency, we then transform the numbers to an equivalent deterministic trend specification in the following way. We stick to the Krueger and Kubler (2006) calibration and only adopt the new correlation structure between TFP innovations and returns. This means that we implicitly compute the average horizon h in the unit root model such that the unconditional variance over h periods coincides with the KK calibration. This gives an average horizon of $h = 19.2751$ years.[12]

---

[11]In earlier versions of this paper we also presented results of such a calibration. Those are available upon request.

[12]Observe that the unit root estimates in fact imply even stronger aggregate fluctuations. Adjusting the variance in the linear trend specification such that the average horizon equals the average horizon of households in our model, appropriately adjusted to account for the correlation of TFP innovations, gives an average horizon of 34.88 years. This implies a standard deviation of 0.039. Relative to the PC calibration this means that the standard deviation of innovations increases by roughly 76 percent. However, the overall effects of this additional increase in risk are small. Results are available upon request.

Overall, targeting the aforementioned four moments with the methods described requires setting $\bar{\zeta} \in \{1.029, 0.971\}$, $\bar{\delta} = 0.11$, $\pi^\zeta = 0.941$ and $\pi^\delta = 0.86$.

In order to check robustness of our findings, we also adopt a standard RBC view of the data and de-trend with the Hodrick-Prescott filter. This yields a highly significant, positive correlation which is comparable in magnitude to our preferred PC (finite difference) calibration.[13]

4.4   Population Data

We assume that agents start working at the biological age of 21, which therefore corresponds to $j = 1$. We set $J = 70$, implying that agents die with certainty at biological age 90, and $jr = 45$, corresponding to a statutory retirement age of 65. Population grows at the rate of 1.1% which reflects the current trend growth of the US population. The conditional survival rates ς_j are imputed from mortality data retrieved from the Human Mortality Database (HMD).

4.5   Household Sector

The value of household's raw time discount factor, β, and the coefficient of relative risk aversion θ are calibrated endogenously (second stage parameters) such that our model produces a capital output ratio of 2.65 and an average equity premium of 0.056.

We determine the intertemporal substitution elasticity as a second-stage parameter such that our model generates a hump-shaped consumption profile. This is achieved via a relatively high value of $\varphi = 1.5$. It is consistent with the range discussed in Bansal and Yaron (2004) and lower than their benchmark value of 2.

The age-specific productivity profile ϵ_j is calibrated to match PSID data applying the method of Huggett (2011).

Our calibration of states $\mathcal{E}_z$ and transition probabilities π^η of the idiosyncratic Markov chain income processes is based on estimates of Storesletten, Telmer, and Yaron (2004b), henceforth STY, for individual wage income processes. STY postulate that the permanent shocks obey an $AR(1)$ process given as

$$\ln(\eta)_{i,j,t} = \rho \ln(\eta)_{i,j-1,t-1} + \epsilon_{i,j,t} \tag{29}$$

---

[13]In earlier versions of this paper we also presented results of such a calibration. Those are available upon request.

where

$$\epsilon_{i,j,t} \sim \mathcal{N}(0, \sigma_t^2) \tag{30}$$

Building on Constantinides and Duffie (1996), STY assume a counter-cyclical, cross-sectional variance of the innovations (CCV). Their estimates are $\rho = 0.952$ and

$$\sigma_t^2 = \begin{cases} \sigma_c^2 = 0.0445 \text{ for } z \in z_1, z_2 \\ \sigma_e^2 = 0.0156 \text{ for } z \in z_3, z_4 \end{cases} \tag{31}$$

where e stands for expansion and c for contraction.

We approximate the above process by discrete two-state Markov process. Denoting state contingency of the innovations by $\sigma^2(z)$, observe that $\sigma(z)_{\ln \eta}^2 = \frac{\sigma^2(z)}{1-\rho^2}$. We then approximate the underlying η by the following symmetric Markov process:

$$\mathcal{E}_z = [\eta_1(z), \eta_2(z)] = [\eta_-(z), \eta_+(z)] \tag{32}$$

$$\pi^\eta = \begin{bmatrix} \bar{\pi}^\eta & 1 - \bar{\pi}^\eta \\ 1 - \bar{\pi}^\eta & \bar{\pi}^\eta \end{bmatrix} \tag{33}$$

$$\Pi = [0.5, 0.5]$$

so that the unconditional mean of the state vector is equal to 1.

Our approximation is different from standard approximations of log income processes in two respects. First, standard approximations do not condition on aggregate states. Second, standard approximations ignore a bias term which gets large when the variance of the estimates increases. We describe the details of our procedure in appendix B.3. Resulting estimates are

$$\eta_1 = \eta_- = \begin{cases} 0.4225 & \text{for } z = z_1, z_2 \\ 0.6196 & \text{for } z = z_3, z_4 \end{cases} \qquad \eta_2 = \eta_+ = \begin{cases} 1.5775 & \text{for } z = z_1, z_2 \\ 1.3804 & \text{for } z = z_3, z_4 \end{cases}$$

and $\bar{\pi}^\eta = 0.9741$.

5   Results

[TBC]

In the discussion of the main results of our quantitative analysis we will refer to the insights derived from the simple model of section 2. In particular, we will highlight the insurance effects against idiosyncratic risk (IR), aggregate risk (AR), and their interaction (LCI) as defined in equation (8), and oppose them with the costs of crowding out.

Calibration of a large and positive correlation of technology shocks and the interest rate, cf. table 1, results in a positive correlation of wages and returns, $cor(w, r) = 0.236$. Volatility of consumption growth in the economy is high, cf. table 3.[14]

Table 3: Model-generated moments (PC)

$cor(w, r)$	$std(\Delta C / C)$
0.236	0.076

The effects of our social security experiment on welfare, capital, and prices are documented in table 4. In the first column (labeled 'GE' for general equilibrium), we compare the two long-run equilibria without any transition. We see that the increase of the contribution rate from $\tau = 0.0$ to $\tau = 0.02$ leads to welfare gains of +3.52%. This number represents the percent of lifetime consumption the agent would be willing to give up to be born into the economy with some social security. It is a very large number. There is substantial crowding out of capital of -5.9%, which leads to the displayed price changes, but this adverse effect is not strong enough to overturn the benefits from insurance.

In order to isolate those insurance benefits, we conduct the partial equilibrium (PE) experiment described in section 3.9. One can think of it as a small open economy, where aggregate prices are determined in the world, and social security is introduced in the small home country. As the second column in table 4 shows, the net welfare gains attributable to the total insurance provided by social security amount to +9.37%. Aggregate prices in this world do not change by construction, and that is why we can isolate the insurance effects. Therefore, the difference between the two welfare numbers $8.88\% - 3.17 = 5.71\%$ can be attributed to the crowding out of capital. Finally, the $\Delta K / K = -29.39\%$ in PE should be interpreted as "less capital being invested abroad": of course agents save less for retirement, and this effect is much smaller in GE because of the mitigating price adjustments.

How much of the welfare gains in partial equilibrium of +8.88% can be attributed to insurance against aggregate risk, how much to idiosyncratic wage risk, how much to CCV, and to survival risk? That is answered in table 5, where we start with an economy with only aggregate risk, then add idiosynratic wage risk on top, then add CVV, and finally also include survival risk. For each economy, we look at the welfare gains from the experiment in

---

[14]We currently work on a new calibration of our model which will bring this number down. This will affect our quantitative but not our qualitative findings on the welfare effects of social security.

Table 4: The social security experiment (PC)

	GE	PE
ΔWelf/Welf	+3.17%	+8.88%
$\Delta K/K$	-5.96%	-29.39%
$\Delta E(r)$	+0.30%	*0.00%*
Δr_f	+0.66%	*0.00%*
$\Delta w/w$	-3.89%	*0.00%*

PE, so that in the last column, we end up with the same +8.88% that we just saw. The first column looks at an economy with only aggregate risk, which therefore is comparable to the partial equilibrium of KK. The welfare gains of in an economy featuring only aggregate risk are +0.97%. We will further decompose this below into the welfare losses in a dynamically efficient risk-free economy and the gains from insurance against aggregate risk.

The second column of table 5 looks at an economy with both aggregate and idiosyncratic risk. Introducing social security in this economy leads to substantially larger welfare gains of +3.38%, which—since this is still PE— are attributable to the intergenerational insurance against aggregate risk plus the intergenerational insurance against idiosyncratic risk. When we add CCV risk, insurance gains go up by another 1.66% (calculated as $5.04\% - 3.38\%$), and looking at the last column we see that adding survival risk adds another 3.84%.

Table 5: Insurance against sources of risk

	aggr. risk	+ idios. wage risk	+ CCV	+ surv. risk
ΔWelf/Welf	+0.97%	+3.38%	+5.04%	+8.88%

Finally, we turn to a decomposition into the insurance components against, AR, IR and their interaction, LCI. This is down with the methods described in section 3.9, especially the system of equations in (27). When solving the economies without risk, we set the average rate of return to 4.2% corresponding to the empirical estimates reported in Siegel (2002). Results of this decomposition analysis are shown in table 6. Out of the total effect of 8.88%, 2.69% ($=dg_c(AR) + dg_c(IR)$) are attributable to the "pure" components of AR and IR. Insurance against the interaction LCI is large, at roughly 1.6%. This makes up about $dg_c(LCI)/dg_c(AR) \cdot 100[\%] = 85\%$ of the components attributable to aggregate risk. Overall, gains from insurance against both interactions is $dg_c(LCI) + dg_c(CCV) = 3.26\%$. This makes up

27

for roughly one third of the total insurance gains.

Table 6: Decomposition of insurance against AR and IR

$g_c(0,0)$	$dg_c(AR)$	$dg_c(IR)$	$dg_c(LCI)$	$dg_c(CCV)$	$dg_c(SR)$
-0.91%	1.88%	0.81%	1.60%	1.66%	3.84%

6   Conclusion

[TBC]

In a life-cycle model, idiosyncratic and aggregate risk interact despite the fact that they are statistically independent. This interaction increases the value of social security. In our general equilibrium analysis, the introduction of a PAYG system leads to strong welfare gains. This stands in contrast to the related literature. The reason for this difference is that in our model, social security provides partial insurance against both idiosyncratic and aggregate risk, as well as their interactions. In fact, the interactions account for one third of the total welfare gains.

In our economy, the intergenerational sharing of aggregate risks is limited to those generations alive at the same point in time. From a social planner's point of view, it would be desirable to share the risk also with future, unborn generations. This could be achieved by allowing the government to take up debt to smooth shocks over time. That would open up an additional insurance channel, which would increase the welfare gains of introducing social security.

We abstract from endogenous labor supply. This biases results in favor of social security for two reasons. First, we do not account for self insurance against risk through endogenous labor supply adjustments. Second, a higher contribution rate would distort labor supply decisions and thereby crowd out aggregate labor supply. We leave an extension of our model along this and other dimensions for future research.

Appendix

A   Proofs

Proof of proposition 1. Maximize

$$E_{t-1}u(c_{i,2,t+1}) = \frac{1}{1-\theta}E_{t-1}\big(\bar{w}_t\big(\bar{R}\eta_{i,1,t}\zeta_t\tilde{\varrho}_{t+1}+\tau\big((1+g)\zeta_{t+1}-\bar{R}\eta_{i,1,t}\zeta_t\tilde{\varrho}_{t+1}\big)\big)\big)^{1-\theta},$$

This is equivalent to maximizing

$$\max \frac{1}{1-\theta}E_{t-1}R_{p,t,t+1}^{1-\theta}$$

where $R_{p,t,t+1} \equiv \eta_{i,1,t}\zeta_t\bar{R}\tilde{\varrho}_{t+1}+\tau\big((1+g)\zeta_{t+1}-\bar{R}\eta_{i,1,t}\zeta_t\tilde{\varrho}_{t+1}\big)$ is a consumption (or portfolio) return. Increasing ex-ante utility for a marginal introduction of social security requires the first-order condition w.r.t. τ to exeed zero, hence:

$$E_{t-1}\left[R_{p,t,t+1}^{-\theta}\frac{\partial R_{p,t,t+1}}{\partial \tau}\right]\big|_{\tau=0} > 0 \tag{34}$$

Evaluated at $\tau = 0$ we have

$$R_{p,t,t+1}^{-\theta}\big|_{\tau=0} = \big(\eta_{i,1,t}\zeta_t\bar{R}\varrho_{t+1}\big)^{-\theta}$$
$$\frac{\partial R_{p,t,t+1}}{\partial \tau}\big|_{\tau=0} = (1+g)\zeta_{t+1}-\eta_{i,1,t}\zeta_t\bar{R}\varrho_{t+1}$$

Equation (34) therefore rewrites as

$$(1+g)E_{t-1}\left[\big(\eta_{i,1,t}\zeta_t\tilde{\varrho}_{t+1}\big)^{-\theta}\zeta_{t+1}\right] > \bar{R}E_{t-1}\left[\big(\eta_{i,1,t}\zeta_t\tilde{\varrho}_{t+1}\big)^{1-\theta}\right]. \tag{35}$$

Rewriting the above and imposing assumption 1 we get equation (4).   □

Proof of proposition 2. Define

$$Z_1 \equiv \big(\eta_{i,1,t}\zeta_t\tilde{\varrho}_{t+1}\big)^{-\theta}\zeta_{t+1}$$
$$Z_2 \equiv \big(\eta_{i,1,t}\zeta_t\tilde{\varrho}_{t+1}\big)^{1-\theta}.$$

By log-normality we have that $EZ_i = \exp(E\ln Z_i + \frac{1}{2}\sigma^2_{\ln Z_i})$, $i = 1, 2$. Observe that

$$E\ln Z_1 = -\theta\left(E\ln\eta_{i,1,t} + E\ln\tilde{\varrho}\right) + (1-\theta)E\ln\zeta$$
$$\sigma^2_{\ln Z_1} = \theta^2\left(\sigma^2_{\ln\eta} + \sigma^2_{\ln\tilde{\varrho}}\right) + (1+\theta^2)\sigma^2_{\ln\zeta}$$

Therefore

$$E_{t-1}[Z_1] = \exp\left(-\theta\left(E\ln\eta_{i,1,t} + \frac{\sigma_{\ln\eta}^2}{2}\right)\right) \cdot \exp\left(-\theta\left(E\ln\tilde{\varrho} + \frac{\sigma_{\ln\tilde{\varrho}}^2}{2}\right)\right) \cdot$$

$$\cdot \exp\left((1-\theta)\left(E\ln\zeta + \frac{\sigma_{\ln\zeta}^2}{2}\right)\right) \exp\left(\frac{1}{2}\theta(1+\theta)\left(\sigma_{\ln\eta}^2 + \sigma_{\ln\tilde{\varrho}}^2 + \sigma_{\ln\zeta}^2\right)\right)$$

$$= (E[\eta_{i,1,t}])^{-\theta}\,(E[\tilde{\varrho}])^{-\theta}\,(E[\zeta])^{1-\theta} \cdot \exp\left(\frac{1}{2}\theta(1+\theta)\left(\sigma_{\ln\eta}^2 + \sigma_{\ln\tilde{\varrho}}^2 + \sigma_{\ln\zeta}^2\right)\right)$$

$$= \exp\left(\frac{1}{2}\theta(1+\theta)\left(\sigma_{\ln\eta}^2 + \sigma_{\ln\tilde{\varrho}}^2 + \sigma_{\ln\zeta}^2\right)\right)$$

whereby the last line follows from assumption 1b.

Next, observe that log-normality implies that

$$\sigma_\eta^2 = var_{t-1}(\eta_{i,1,t}) = \exp\left(2E\ln\eta_{i,1,t} + \sigma_{\ln\eta}^2\right)\left(\exp\left(\sigma_{\ln\eta}^2\right) - 1\right)$$

$$= (E\eta_{i,1,t})^2\left(\exp\left(\sigma_{\ln\eta}^2\right) - 1\right)$$

$$= \left(\exp\left(\sigma_{\ln\eta}^2\right) - 1\right)$$

whereby the last line again follows from assumption 1b. Hence:

$$\sigma_{\ln\eta}^2 = \ln\left(1 + \sigma_\eta^2\right)$$

with corresponding expressions for $\sigma_{\ln\zeta}^2$ and $\sigma_{\ln\tilde{\varrho}}^2$. Therefore:

$$\exp\left(\frac{1}{2}\theta(1+\theta)\left(\sigma_{\ln\eta}^2 + \sigma_{\ln\tilde{\varrho}}^2 + \sigma_{\ln\zeta}^2\right)\right) = \left((1+\sigma_\eta^2)(1+\sigma_\zeta^2)(1+\sigma_{\tilde{\varrho}}^2)\right)^{\frac{1}{2}\theta(1+\theta)}$$

We consequently have

$$E_{t-1}[Z_1] = \left((1+\sigma_\eta^2)(1+\sigma_\zeta^2)(1+\sigma_{\tilde{\varrho}}^2)\right)^{\frac{1}{2}\theta(1+\theta)}$$

As to $E_{t-1}[Z_2]$ observe that

$$E\ln Z_2 = (1-\theta)\left(E\ln\eta_{i,1,t} + E\ln\zeta + E\ln\tilde{\varrho}\right)$$
$$\sigma_{\ln Z_2}^2 = (1-\theta)^2\left(\sigma_{\ln\eta}^2 + \sigma_{\ln\zeta}^2 + \sigma_{\ln\tilde{\varrho}}^2\right)$$

Therefore

$$E_{t-1}[Z_2] = \exp\left((1-\theta)\left(E\ln\eta_{i,1,t} + \frac{\sigma_{\ln\eta}^2}{2}\right)\left(E\ln\tilde{\varrho} + \frac{\sigma_{\ln\tilde{\varrho}}^2}{2}\right)\left(E\ln\zeta + \frac{\sigma_{\ln\zeta}^2}{2}\right)\right)$$

$$\cdot \exp\left(\frac{1}{2}\theta(\theta-1)\left(\sigma_{\ln\eta}^2 + \sigma_{\ln\tilde{\varrho}}^2 + \sigma_{\ln\zeta}^2\right)\right)$$

$$= \left((1+\sigma_\eta^2)(1+\sigma_\zeta^2)(1+\sigma_{\tilde{\varrho}}^2)\right)^{\frac{1}{2}\theta(\theta-1)}$$

Hence

$$\frac{E_{t-1}[Z_1]}{E_{t-1}[Z_2]} = \left((1+\sigma_\eta^2)(1+\sigma_\zeta^2)(1+\sigma_{\hat{\varrho}}^2)\right)^\theta$$

$\square$

Illustration 1. *Let us provide a simplified numerical illustration. Below, we calibrate our model with an annual income processes given by $\ln(\eta_{i,j,t}) = \rho \ln(\eta_{i,j-1,t-1}) + \epsilon_{i,j,t}$ where j is actual age of a working household, t is time, $\epsilon_{i,j,t} \sim \mathcal{N}(0, \sigma_t^2)$, hence $\eta_{i,j,t}$ is distributed as log-normal for all i, j, t and ρ is the autocorrelation coefficient. While we consider time variation in variances below, let us assume constant variances for now. Our calibration has an average variance of $\sigma^2 \approx 0.03$. We also calibrate $\rho = 0.952$. Consider the overall variance of income risk at retirement, that is, after a period in the work force of about 45 years. For AR(1) processes with such a long horizon, the approximate infinite horizon[15] formula to compute the variance of $\ln \eta_{i,1,t}$ at retirement is given by $\frac{1}{1-\rho^2}\sigma_\epsilon^2$. Using our numbers we accordingly have that the variance of $\ln \eta_{i,1,t}$ at retirement is given by $\frac{1}{1-0.952^2} \cdot 0.03 = 10.67 \cdot 0.03$. By the formula for log-normal random variables, the variance of $\eta_{i,1,t}$ at retirement is therefore $var(\eta_{i,1,t}) = (E[\eta_{i,1,t}])^2 \left(\exp\left(\sigma_{\ln \eta}^2\right) - 1\right) = \exp\left(10.67 \cdot 0.03\right) - 1 = 0.37$.[16]*

Derivation of equation 8. We want to evaluate CEV between two scenarios, i.e., comparing $E_{t-1}u(c_{i,2,t+1_{\tau>0}})$ with $E_{t-1}u(c_{i,2,t+1_{\tau=0}})$. To simplify, let us use that

$$E_{t-1}u(c_{i,2,t+1_{\tau>0}}) = E_{t-1}u(c_{i,2,t+1_{\tau=0}}) + \frac{\partial E_{t-1}u(c_{i,2,t+1_{\tau=0}})}{\partial \tau}d\tau.$$

---

[15]The exact formula is $\frac{1-\rho^{2(j_r-1)}}{1-\rho^2}$ where j_r is the retirement age but the term $\rho^{2(j_r-1)}$ is negligible.

[16]As we describe in our main text, our estimates are based on Storesletten, Telmer, and Yaron (2004a) who use after tax earnings data and control for aggregate fluctuations. Observe that these numbers are a conservative estimate of the overall dispersion of earnings inequalities at retirement because we ignore the dispersion of skills and learning abilities at the beginning of the life-cycle. The more recent work by Huggett, Ventrua, and Yaron (2011) attributes about 60 of the overall variation in life-time income to variations in initial conditions. However, it is rather education policies than pension policies and social insurance that should target such differences. Huggett et al. (2011)'s specification for income shocks is a unit root process. Their estimate of the standard deviation of the innovation of this process is 0.111. This would roughly double the relevance of the interaction term at retirement to 0.74. However, the estimates of Huggett et al. (2011) are based on pre-tax earnings data and the authors do not control for the business cycle. This may explain these substantial differences.

and evaluate this expression at $\tau = 0$.

We have that, evaluated at $\tau = 0$,

$$
\begin{aligned}
\frac{\partial E_{t-1} u(c_{i,2,t+1_{\tau=0}})}{\partial \tau} &= \bar{w}_t^{1-\theta} E_{t-1} \left[\left(\bar{R} \eta \zeta_t \tilde{\varrho}_{t+1} \right)^{-\theta} \cdot \left((1+g)\zeta_{t+1} - \bar{R} \eta \zeta_t \tilde{\varrho}_{t+1} \right) \right] \\
&= \bar{w}_t^{1-\theta} \left(\bar{R}^{-\theta}(1+g) E_{t-1} \left[(\eta \zeta_t \tilde{\varrho}_{t+1})^{-\theta} \zeta_{t+1} \right] - \bar{R}^{1-\theta} E_{t-1} \left[(\eta \zeta_t \tilde{\varrho}_{t+1})^{1-\theta} \right] \right) \\
&= \bar{w}_t^{1-\theta} \bar{R}^{1-\theta} \left(\frac{1+g}{\bar{R}} E_{t-1} Z_1 - E_{t-1} Z_2 \right)
\end{aligned}
$$

where Z_1, Z_2 are defined in our proof to proposition 2.

We also have that

$$
\begin{aligned}
E_{t-1} u(c_{i,2,t+1_{\tau=0}}) &= \frac{1}{1-\theta} \bar{w}_t^{1-\theta} \bar{R}^{1-\theta} E_{t-1} \left(\eta_{i,1,t} \zeta_t \tilde{\varrho}_{t+1} \right)^{1-\theta} \\
&= \frac{1}{1-\theta} \bar{w}_t^{1-\theta} \bar{R}^{1-\theta} E_{t-1} Z_2.
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
E_{t-1} u(c_{i,2,t+1_{\tau>0}}) &= \frac{1}{1-\theta} \bar{w}_t^{1-\theta} \bar{R}^{1-\theta} E_{t-1} Z_2 \\
&\quad + \bar{w}_t^{1-\theta} \bar{R}^{1-\theta} \left(\frac{1+g}{\bar{R}} E_{t-1} Z_1 - E_{t-1} Z_2 \right) d\tau.
\end{aligned}
$$

The CEV, denoted by g_c, is defined by the relationship:

$$
E_{t-1} u(c_{i,2,t+1_{\tau=0}}(1+g_c)) = E_{t-1} u(c_{i,2,t+1_{\tau>0}}),
$$

from which, using the above formulae, we get

$$
\begin{aligned}
(1+g_c)^{1-\theta} \frac{1}{1-\theta} \bar{w}_t^{1-\theta} \bar{R}^{1-\theta} E_{t-1} Z_2 &= \frac{1}{1-\theta} \bar{w}_t^{1-\theta} \bar{R}^{1-\theta} E_{t-1} Z_2 \\
&\quad + \bar{w}_t^{1-\theta} \bar{R}^{1-\theta} \left(\frac{1+g}{\bar{R}} E_{t-1} Z_1 - E_{t-1} Z_2 \right) d\tau.
\end{aligned}
$$

Hence:

$$
\begin{aligned}
(1+g_c)^{1-\theta} &= 1 + \frac{\bar{w}_t^{1-\theta} \bar{R}^{1-\theta} \left(\frac{1+g}{\bar{R}} E_{t-1} Z_1 - E_{t-1} Z_2 \right)}{\frac{1}{1-\theta} \bar{w}_t^{1-\theta} \bar{R}^{1-\theta} E_{t-1} Z_2} d\tau \\
&= 1 + (1-\theta) \left(\frac{1+g}{\bar{R}} \frac{E_{t-1} Z_1}{E_{t-1} Z_2} - 1 \right) d\tau \\
&= 1 + (1-\theta) \left(\frac{1+g}{\bar{R}} (1+V)^\theta - 1 \right) d\tau
\end{aligned}
$$

where the last line again follows from the proof to proposition 2.

Hence,

$$g_c = \left(1 + (1 - \theta)\left(\frac{1+g}{\bar{R}}(1+V)^\theta - 1\right)d\tau\right)^{\frac{1}{1-\theta}} - 1.$$

or, expressed in logs, i.e., $g_c \approx \ln(1 + g_c)$, we get

$$g_c \approx \frac{1}{1-\theta} \cdot \ln\left(1 + (1-\theta)\left(\frac{1+g}{\bar{R}}(1+V)^\theta - 1\right)d\tau\right)$$

$$\approx \left(\frac{1+g}{\bar{R}}(1+V)^\theta - 1\right)d\tau$$

Taking a first-order Taylor series expansion of the above round $V = 0$ we get

$$g_c \approx \left(\frac{1+g}{\bar{R}} - 1 + \theta\frac{1+g}{\bar{R}}V\right) \cdot d\tau$$

The first term in brackets is the deterministic part. The second term is the additional gain due to risk which is linear in V. $\qquad\square$

Proof of proposition 3. To establish proposition 3a we have to show that

$$\frac{1}{\zeta_l}E_{t-1}\frac{1}{\eta_{i,1,l}} + \frac{1}{\zeta_h}E_{t-1}\frac{1}{\eta_{i,1,h}} > \left(\frac{1}{\zeta_l} + \frac{1}{\zeta_h}\right)E_{t-1}\frac{1}{\eta_{i,1,t}}$$

$$\Leftrightarrow \quad \frac{1}{\zeta_l}\left(E_{t-1}\frac{1}{\eta_{i,1,l}} - E_{t-1}\frac{1}{\eta_{i,1,t}}\right) + \frac{1}{\zeta_h}\left(E_{t-1}\frac{1}{\eta_{i,1,h}} - E_{t-1}\frac{1}{\eta_{i,1,t}}\right) > 0. \quad (36)$$

Under assumption 3 we have that

$$E_{t-1}\frac{1}{\eta_{i,1,t}} = \exp\left(-\left(E\ln\eta + \frac{1}{2}\sigma_{\ln\eta}^2\right)\right)$$

$$E_{t-1}\frac{1}{\eta_{i,1,l}} = \exp\left(-\left(E\ln\eta + \frac{1}{2}\left(\sigma_{\ln\eta}^2 - \Delta\right)\right)\right)$$

$$E_{t-1}\frac{1}{\eta_{i,1,h}} = \exp\left(-\left(E\ln\eta + \frac{1}{2}\left(\sigma_{\ln\eta}^2 + \Delta\right)\right)\right).$$

33

Therefore

$$E_{t-1}\frac{1}{\eta_{i,1,l}} - E_{t-1}\frac{1}{\eta_{i,1,t}}$$

$$= \exp\left(-\left(E\ln\eta + \frac{1}{2}\left(\sigma_{\ln\eta}^2 - \Delta\right)\right)\right) - \exp\left(-\left(E\ln\eta + \frac{1}{2}\sigma_{\ln\eta}^2\right)\right)$$

$$= \exp\left(-\left(E\ln\eta + \frac{1}{2}\sigma_{\ln\eta}^2\right)\right)\exp\left(\frac{1}{2}\Delta\right) - \exp\left(-\left(E\ln\eta + \frac{1}{2}\sigma_{\ln\eta}^2\right)\right)$$

$$= \exp\left(-\left(E\ln\eta + \frac{1}{2}\sigma_{\ln\eta}^2\right)\right)\left(\exp\left(\frac{1}{2}\Delta\right) - 1\right)$$

$$E_{t-1}\frac{1}{\eta_{i,1,h}} - E_{t-1}\frac{1}{\eta_{i,1,t}}$$

$$= \exp\left(-\left(E\ln\eta + \frac{1}{2}\left(\sigma_{\ln\eta}^2 + \Delta\right)\right)\right) - \exp\left(-\left(E\ln\eta + \frac{1}{2}\sigma_{\ln\eta}^2\right)\right)$$

$$= \exp\left(-\left(E\ln\eta + \frac{1}{2}\sigma_{\ln\eta}^2\right)\right)\exp\left(-\frac{1}{2}\Delta\right) - \exp\left(-\left(E\ln\eta + \frac{1}{2}\sigma_{\ln\eta}^2\right)\right)$$

$$= \exp\left(-\left(E\ln\eta + \frac{1}{2}\sigma_{\ln\eta}^2\right)\right)\left(\exp\left(-\frac{1}{2}\Delta\right) - 1\right)$$

Equation (36) therefore rewrites as

$$\frac{1}{\zeta_l}\left(\exp\left(\frac{1}{2}\Delta\right) - 1\right) + \frac{1}{\zeta_h}\left(\exp\left(-\frac{1}{2}\Delta\right) - 1\right) > 0. \qquad (37)$$

Observe that $\exp\left(-\frac{1}{2}\Delta\right) - 1 < 0$, $\exp\left(\frac{1}{2}\Delta\right) - 1 > 0$ and convexity of the exponential function implies that

$$|\exp\left(-\frac{1}{2}\Delta\right) - 1| < |\exp\left(\frac{1}{2}\Delta\right) - 1|.$$

Therefore

$$\frac{1}{\zeta_l}\left(\exp\left(\frac{1}{2}\Delta\right) - 1\right) + \frac{1}{\zeta_h}\left(\exp\left(-\frac{1}{2}\Delta\right) - 1\right)$$

$$> \frac{1}{\zeta_l}\left(\exp\left(\frac{1}{2}\Delta\right) - 1\right) - \frac{1}{\zeta_h}\left(\exp\left(\frac{1}{2}\Delta\right) - 1\right)$$

$$= \left(\exp\left(\frac{1}{2}\Delta\right) - 1\right)\left(\frac{1}{\zeta_l} - \frac{1}{\zeta_h}\right)$$

$$= \left(\exp\left(\frac{1}{2}\Delta\right) - 1\right)\left(\frac{1}{\zeta_l} - \frac{1}{2 - \zeta_l}\right)$$

$$= \left(\exp\left(\frac{1}{2}\Delta\right) - 1\right)\frac{2(1 - \zeta_l)}{\zeta_l(2 - \zeta_l)} > 0.$$

To establish proposition 3b use assumption 3 to rewrite equation (37) as

$$f(\Delta_\zeta) \equiv \frac{1}{1 - \Delta_\zeta} \left(\exp\left(\frac{1}{2}\Delta\right) - 1 \right) + \frac{1}{1 + \Delta_\zeta} \left(\exp\left(-\frac{1}{2}\Delta\right) - 1 \right) > 0.$$

Observe that

$$\frac{\partial f(\Delta_\zeta)}{\partial \Delta_\zeta} = \left(\frac{1}{1 - \Delta_\zeta}\right)^2 \left(\exp\left(\frac{1}{2}\Delta\right) - 1 \right) - \left(\frac{1}{1 + \Delta_\zeta}\right)^2 \left(\exp\left(-\frac{1}{2}\Delta\right) - 1 \right)$$

$$= \left(\frac{1}{1 - \Delta_\zeta}\right)^2 \underbrace{\left(\exp\left(\frac{1}{2}\Delta\right) - 1 \right)}_{>0} + \left(\frac{1}{1 + \Delta_\zeta}\right)^2 \underbrace{\left(1 - \exp\left(-\frac{1}{2}\Delta\right) \right)}_{>0}$$

$$> 0$$

Hence, a mean preserving spread of ζ increases the effect of CCV. $\qquad\square$

B Computational Solution

B.1 Aggregate Problem

In order to compute the stationary competitive equilibrium of our model, we apply the Krusell and Smith (1997) method. Specifically, we follow Storesletten, Telmer, and Yaron (2007) (STY) and approximate the aggregate law of motion as

$$(k', \mu') = \hat{H}(t; k, \mu, z, z') \tag{38}$$

where k is the capital stock per efficiency unit and $\mu = \mathbb{E}r' - r^{f'}$ is the equity premium. That is, we approximate the distribution Φ by two "moments" where the equity premium captures information about equity and bond holding moments. Our approach differs from STY in three ways: (i) we plan to explicitly compute transitional dynamics between two stationary competitive equilibria (which fluctuate in two ergodic sets), (ii) we do not use simulation techniques to aggregate on the idiosyncratic states of the distribution and (iii) we compute an approximate equilibrium, referred to as a "mean shock equilibrium", which serves three purposes: first, it enables us to initialize the cross-sectional distribution of agents second, we use it in order to calibrate our model in the initial competitive equilibrium in all periods $t \leq 0$ (for $\tau_t = \tau_0$) and third, it determines the means of the aggregate grids which we employ in the stochastic solution of our model. Computation of the "mean shock equilibrium" is by standard methods to solve OLG models without any aggregate risk. But in contrast to fully deterministic models, the mean shock equilibrium gives rise to an equity premium.

B.1.1   Mean Shock Equilibrium

As an initialization step, we solve for a degenerate path of the economy where the realizations of all aggregate shocks are at their respective means. We accordingly set $z = \bar{z} = \mathbb{E}z$ and $\delta = \bar{\delta} = \mathbb{E}\delta$. We assume that households accurately solve their forecasting problem for each realization of the aggregate state. This means that we approximate the above approximate law of motion as

$$(k', \mu') = \hat{\hat{H}}(k, \mu, \bar{z}, \bar{z}') \tag{39}$$

Observe that in the two stationary equilibria of our model, we have that fixed point relation

$$(k', \mu') = \hat{\hat{H}}(k, \mu, \bar{z}, \bar{z}') = (k, \mu) \tag{40}$$

With these assumptions, we can solve the mean shock path by standard Gauss-Seidel iterations as, e.g., described in Auerbach and Kotlikoff (1987). We adopt the modifications described in Ludwig (2007). While the numerical methods are the same as in the solution to a deterministic economy, the actual behavior of households fully takes into account the stochastic nature of the model. This also means that we solve the household problem using recursive methods and store the solutions to the household problem on grids of the idiosyncratic state x. The fixed-point computed in this auxiliary equilibrium gives k^{ms} and μ^{ms} as aggregate moments and cross-sectional distributions of agents as induced by the mean shock path. We denote these distributions by Φ^{ms}.

B.1.2   Grids

To construct the grids for the the aggregate states k and μ, $\mathcal{G}^k$, $\mathcal{G}^\mu$, define scaling factors s^k and s^μ and the number of grid points, n. We set $s^k = 0.8$ $s^\mu = 0.6$, and $n = 7$. Using these factors, we construct symmetric grids around k^{ms}, μ^{ms}.

B.1.3   Stochastic Solution

In order to solve for the stochastic recursive equilibria of our model, we use simulation methods. To this end, we specify the approximate law of motion in (38) as:

$$\ln(k_{t+1}) = \psi_0^k(z) + \psi_1^k(z)\ln(k_t) + \psi_2^k(z)\ln(\mu_t) \tag{41a}$$

$$\ln(\mu_{t+1}) = \psi_0^\mu(z) + \psi_1^\mu(z)\ln(k_{t+1}) + \psi_2^\mu(z)\ln(\mu_t) \tag{41b}$$

Like in Krusell and Smith (1997), the forecast for k_{t+1} is used to forecast μ_{t+1}. Intuitively, k_{t+1} contains a lot of information on the savings choice of the agent and therefore on the returns next period. Note that, in each period, μ_t is an "endogenous state", the realization of which has to be pinned down in that particular period (in contrast to k_t which is given in period t from decisions $t-1$). As in the standard application of the Krusell and Smith (1998) method, the coefficients also depend on the realization of the aggregate state, z.

Stationary Equilibria

Define a number M of stochastic simulations and a number of $s < M$ of simulations to be discarded. We follow GM and set $M = 5500$ and $s = 500$. Also define a tolerance ζ. Further, draw a sequence for z for periods $t = -M, \ldots, 0$ and denote these realizations by $z_{-M}, \ldots, z_0$. Notice that we thereby use the same sequence of aggregate shocks (as given by a random number generator) in each iteration. Collecting coefficients as $\Psi = [\psi_0^k, \psi_1^k, \psi_2^k, \psi_0^\mu, \psi_1^\mu, \psi_2^\mu]'$, the iteration is as follows:

1. Initialization: Guess Ψ.

2. In each iteration i do the following:

   (a) Solution of household problem. We store the solutions of the household problem on the $\mathcal{G}^{hh} = \mathcal{G}^J \times \mathcal{G}^x \times \mathcal{G}^z \times \mathcal{G}^k \times \mathcal{G}^\mu$. This gives us policy functions for all households, e.g., $c(j, x; z, k, \mu)$, $\kappa(j, x; z, k, \mu)$, $a'(j, x; z, k, \mu)$.

   (b) Simulation and aggregation. We simulate the model economy for the M realizations of aggregate shocks, z. To aggregate on the idiosyncratic states, we start in period $t = -M$ with the initial distribution generated by the mean shock path, Φ^{ms}. We then loop forward using the transition functions Q to update distributions. Notice that, conditional on the realization of z, this aggregation is by standard methods that are used in OLG models with idiosyncratic risk. Simulation and aggregation then gives us M realizations of k_t and μ_t for $t = -M, \ldots, 0$. Observe that, in order to compute the realizations for μ_t, we have to solve for the bond market clearing equilibrium in each t. We do so by using a univariate function solver (Brent's method). We are thereby more accurate than Gomes and Michaelides (2008) who simply interpolate on $\mathcal{G}^\mu$.

(c) From the stochastic simulations, discard the first s observations and, for the remaining periods $t = s, \ldots, 0$ run regressions on:

$$\ln(k_{t+1}) = \tilde{\psi}_0^k(z) + \tilde{\psi}_1^k(z)\ln(k_t) + \tilde{\psi}_2^k(z)\ln(\mu_t) + \vartheta_{t+1}^k \quad (42a)$$

$$\ln(\mu_{t+1}) = \tilde{\psi}_0^\mu(z) + \tilde{\psi}_1^\mu(z)\ln(k_t) + \tilde{\psi}_2^\mu(z)\ln(\mu_t) + \vartheta_{t+1}^\mu \quad (42b)$$

and collect the resulting coefficient estimates in the vector $\tilde{\Psi}$.

(d) IF $\|\Psi_i - \tilde{\Psi}_i\| < \zeta$ then STOP, ELSE define

$$g(\Psi) = \Psi - \tilde{\Psi}(\Psi) \quad (43)$$

as the distance function (=root finding problem) and update Ψ_{i+1} as

$$\Psi_{i+1} = \Psi_i - sJ(\Psi)^{-1}g(\Psi) \quad (44)$$

where $J(\Psi)$ is the Jacobi matrix of the system of equations in (43) and s is a scaling factor. Continue with step 2a. We solve the root finding problem using Broyden's method, see Ludwig (2007).

B.2 Household Problem

We iterate on the Euler equation, using ideas developed in Carroll (2006). As derived in section 3.7, the transformed dynamic programming problem of the household reads as

$$u(t, j, x, \eta; z, k, \mu)$$
$$= \max_{c, \kappa, a'} \left\{ \left[c^{\frac{1-\theta}{\gamma}} + \tilde{\beta} \left(\mathbb{E}\left[u\left(t+1, j+1, x', \eta'; z', k', \mu'\right)^{1-\theta} \right] \right)^{\frac{1}{\gamma}} \right]^{\frac{\gamma}{1-\theta}} \right\}$$
$$\text{s.t.} \quad x = a' + c,$$

where $x' = a'\tilde{R}' + y'$, with $\tilde{R}' = \frac{(1+r^{f'}+\kappa(r'-r^{f'}))}{(1+g)}$, and $\tilde{\beta} = \beta\varsigma_{j+1}^{\frac{1}{\gamma}}((1+g))^{\frac{1-\theta}{\gamma}}$. Dropping the time index to simplify notation and using the dynamic budget constraint in the continuation value we get

$$u(j, \cdot) = \max_{c, \kappa} \left\{ \left[c^{\frac{1-\theta}{\gamma}} + \tilde{\beta} \left(\mathbb{E}\left[u(j+1, (x-c)\tilde{R}' + y', \cdot)^{1-\theta} \right] \right)^{\frac{1}{\gamma}} \right]^{\frac{\gamma}{1-\theta}} \right\} \quad (45)$$

The first-order conditions are given by:

$$c: \quad c^{\frac{1-\theta-\gamma}{\gamma}} - \tilde{\beta} \left(\mathbb{E}\left[u(j+1, \cdot)^{1-\theta} \right] \right)^{\frac{1-\gamma}{\gamma}}$$
$$\cdot \mathbb{E}\left[u(j+1, \cdot)^{-\theta} u_{x'}(j+1, \cdot)\tilde{R}' \right] = 0 \quad (46a)$$

$$\kappa: \quad \mathbb{E}\left[u(j+1, \cdot)^{-\theta} u_{x'}(j+1, \cdot)\left(r' - r^{f'}\right) \right] = 0 \quad (46b)$$

and the envelope condition reads as:

$$u_x = \left(c^{\frac{1-\theta}{\gamma}} + \tilde{\beta} \left(\mathbb{E} \left[u(j+1,\cdot)^{1-\theta} \right] \right)^{\frac{1}{\gamma}} \right)^{\frac{\gamma-1+\theta}{1-\theta}} \cdot \ldots$$

$$\cdot \tilde{\beta} \left(\mathbb{E} \left[u(j+1,\cdot)^{1-\theta} \right] \right)^{\frac{1-\gamma}{\gamma}} \mathbb{E} \left[\tilde{R}' u(j+1,\cdot)^{-\theta} u_{x'}(j+1,\cdot) \right]$$

$$= u(j,\cdot)^{\frac{\gamma-1+\theta}{\gamma}} \tilde{\beta} \left(\mathbb{E} \left[u(j+1,\cdot)^{1-\theta} \right] \right)^{\frac{1-\gamma}{\gamma}} \cdot \ldots$$

$$\cdot \mathbb{E} \left[\tilde{R}' u(j+1,\cdot)^{-\theta} u_{x'}(j+1,\cdot) \right]$$

$$= \left(\frac{c}{u(j,\cdot)} \right)^{\frac{1-\theta-\gamma}{\gamma}} , \tag{47}$$

where the last line follows from equation (46a) and is exactly the result one would get from direct application of the Benveniste/Scheinkman theorem to recursive preferences, namely $v_x = u_1(c, Ev)$ (see Weil (1989)). Plugging this into the FOCs we get

$$c: \quad c^{\frac{1-\theta-\gamma}{\gamma}} - \tilde{\beta} \left(\mathbb{E} \left[u(j+1,\cdot)^{1-\theta} \right] \right)^{\frac{1-\gamma}{\gamma}} \ldots$$

$$\cdot \mathbb{E} \left[u(j+1,\cdot)^{\frac{(1-\theta)(\gamma-1)}{\gamma}} (c')^{\frac{1-\theta-\gamma}{\gamma}} \tilde{R}' \right] = 0 \tag{48a}$$

$$\kappa: \quad \mathbb{E} \left[u(j+1,\cdot)^{\frac{(1-\theta)(\gamma-1)}{\gamma}} (c')^{\frac{1-\theta-\gamma}{\gamma}} \left(r' - r^{f'} \right) \right] = 0 \tag{48b}$$

With respect to our numerical solution, we will interpolate the functions $u(j,\cdot)$ and $c(j,\cdot)$. Note that we can expect $u(j,\cdot)$ to be approximately linear, since in period J it is simply given by $u(J) = c_J = x_J$.

Next, notice that $u(j+1,\cdot)$ and c' are functions of $(x-c)$ so that c shows up on both sides of the equation in (48a). This would require calling a nonlinear solver whenever we solve optimal consumption and portfolio shares. To alleviate this computational burden we employ the endogenous grid method of Carroll (2006). Accordingly, instead of working with an *exogenous* grid for x (and thereby an *endogenous* grid for savings, $s = x - c$) we revert the order and work with an *exogenous* grid for $s = x - c$ and an *endogenous* grid for x.

So, roughly speaking, for each age j and each grid point in the savings grid $\mathcal{G}^s$, our procedure is the following:

1. Solve equation (48b) for κ using a univariate equation solver (Brent's method).

2. Given the solution to (48b) invert (48a) to compute

$$c = \left(\tilde{\beta} \left(\mathbb{E} \left[u(j+1,\cdot)^{1-\theta} \right] \right)^{\frac{1-\gamma}{\gamma}} \mathbb{E} \left[v(j+1,\cdot)^{-\theta} \tilde{R}' \right] \right)^{\frac{\gamma}{1-\theta-\gamma}} . \tag{49}$$

3. Update u, u_x and v.

More precisely, our procedure is as follows:

1. Loop on the grids of the aggregate states, $\mathcal{G}^z$, $\mathcal{G}^k$, $\mathcal{G}^\mu$.

2. For each (z, k, μ) use (41) to compute the associated k', μ'.

3. Initialize the loop on age for $j = J$ by setting $c_J = x_J$ and compute $u(x_J) = c_J$, $u_x(x_J) = 1$ and $v(x_J) = u(x_J) = c_J$.

4. Loop backwards in age from $j = J - 1, \ldots, 0$ as follows:

    (a) As $k' \notin \mathcal{G}^k$, $\mu' \notin \mathcal{G}^\mu$, interpolate on the aggregate states and store the interpolated objects $u(k', \mu', x', z', j+1)$, $v(k', \mu', x', z', j+1)$ as a projection on $\mathcal{G}^x$. Do so for each $z' \in \mathcal{G}^z$. Denote the interpolated objects as $\bar{u}(x', z', j+1)$, $\bar{v}(x', z', j+1)$.

    (b) For each s in $\mathcal{G}^s$ first solve (48b) for κ. To so, we have to loop on $z' \in \mathcal{G}^z$ (as well as the idiosyncratic shock) to evaluate the expectation taking into account the Markov transition matrix $\pi(z'|z)$. In this step, we also use the law of motion of the idiosyncratic state x:

$$x' = s\tilde{R}' + \mathrm{y}'$$

(50)

    As, generally, $x' \notin \mathcal{G}^x$ we have to interpolate on $\bar{u}(x', z', j+1)$ and $\bar{v}(x', z', j+1)$ before evaluating the expectation.

    (c) Taking the optimal κ as given, next compute c from (49). Again we interpolate on $\bar{u}(x', z', j+1)$ and $\bar{v}(x', z', j+1)$ before evaluating the expectation.

B.3   Calibration of Income Process

We determine $\eta\mp(z)$ and $\bar{\pi}^\eta$ such that we match the unconditional variance of the STY estimates, i.e.,

$$E[(\ln \eta')^2 \mid z'] = \sigma(z')_{\ln \eta}^2$$

(51)

and the unconditional autocorrelation, i.e.,

$$\frac{E[\ln \eta' \ln \eta]}{E[(\ln \eta')^2]} = \rho.$$

(52)

To match the variance we specify the states of the Markov process as

$$\eta_{\mp}(z) = \frac{2\exp(1 \mp \tilde{\sigma}(z))}{\exp(1 - \tilde{\sigma}(z)) + \exp(1 + \tilde{\sigma}(z))}$$

so that the unconditional mean equals one.

We pick $\tilde{\sigma}(z)$ such that the unconditional variance—which of course preserves its contingency on z—satisfies (51). To achieve this, observe that

$$\ln \eta_{\mp} = \ln\left(\frac{2}{\exp(1 - \tilde{\sigma}(z)) + \exp(1 + \tilde{\sigma}(z))}\right) + 1 \mp \tilde{\sigma}(z) \equiv \phi(\tilde{\sigma}(z)) \mp \tilde{\sigma}(z).$$

Hence

$$E[(\ln \eta')^2 \mid \eta = \eta_-, z'] = \bar{\pi}^\eta(\phi(\tilde{\sigma}(z')) - \tilde{\sigma}(z'))^2 +$$
$$(1 - \bar{\pi}^\eta)(\phi(\tilde{\sigma}(z')) + \tilde{\sigma}(z'))^2$$
$$E[(\ln \eta')^2 \mid \eta = \eta_+, z'] = \bar{\pi}^\eta(\phi(\tilde{\sigma}(z')) + \tilde{\sigma}(z'))^2 +$$
$$(1 - \bar{\pi}^\eta)(\phi(\tilde{\sigma}(z')) - \tilde{\sigma}(z'))^2.$$

The unconditional mean of the above—conditional on z'—is

$$E[(\ln \eta')^2 \mid z'] = \phi(\tilde{\sigma}(z'))^2 + \tilde{\sigma}(z')^2.$$

To determine $\tilde{\sigma}(z)$, we then numerically solve the distance function

$$f(\tilde{\sigma}(z)) = \phi(\tilde{\sigma}(z))^2 + \tilde{\sigma}(z)^2 - \sigma(z)_{\ln \eta}^2 = 0$$

for all z. Standard procedures ignore the bias term $\phi(\tilde{\sigma}(z))^2$.

To determine $\bar{\pi}^\eta$ observe that, in the stationary invariant distribution, we have

$$E[\ln \eta' \ln \eta \mid \eta = \eta_-] = \sum_z \Pi^z(z) \left\{ \bar{\pi}^\eta(\phi(\tilde{\sigma}(z)) - \tilde{\sigma}(z))^2 + \right.$$
$$\left. (1 - \bar{\pi}^\eta)(\phi(\tilde{\sigma}(z)) - \tilde{\sigma}(z))(\phi(\tilde{\sigma}(z)) + \tilde{\sigma}(z)) \right\}$$
$$E[\ln \eta' \ln \eta \mid \eta = \eta_+] = \sum_z \Pi^z(z) \left\{ \bar{\pi}^\eta(\phi(\tilde{\sigma}(z)) + \tilde{\sigma}(z))^2 + \right.$$
$$\left. (1 - \bar{\pi}^\eta)(\phi(\tilde{\sigma}(z)) + \tilde{\sigma}(z))(\phi(\tilde{\sigma}(z)) - \tilde{\sigma}(z)) \right\}$$

and

$$E[\ln \eta' \ln \eta] = \sum_\eta \Pi^\eta(\eta) E[\ln \eta' \ln \eta \mid \eta]$$

as well as

$$E[(\ln \eta')^2] = \sum_{z'} \Pi^{z'} E[(\ln \eta')^2 \mid z'].$$

Noticing that

$$\frac{E[\ln \eta' \ln \eta]}{E[(\ln \eta')^2]} = \rho$$

we use the above relationships to determine $\bar{\pi}^\eta$.

References

Auerbach, A. J. and L. J. Kotlikoff (1987). *Dynamic Fiscal Policy*. Cambridge, MA: Cambridge University Press.

Ball, L. and N. G. Mankiw (2007). Intergenerational risk sharing in the spirit of arrow, debreu, and rawls, with applications to social security design. *Journal of Political Economy 115*(4), 523–547.

Bansal, R. and A. Yaron (2004). Risks for the long run: A potential resolution of asset pricing puzzles. *The Journal of Finance 59*(4), 1481–1509.

Bohn, H. (2001). Social Security and Demographic Uncertainty: The Risk Sharing Properties of Alternative Policies. In J. Campbell and M. F. Feldstein (Eds.), *Risk Aspects of Investment Based Social Security Reform*, pp. 203–241. University of Chicago Press.

Bohn, H. (2009). Intergenerational Risk Sharing and Fiscal Policy. *Journal of Monetary Economics 56*, 805–816.

Carroll, C. D. (2006, June). The method of endogenous gridpoints for solving dynamic stochastic optimization problems. *Economics Letters 91*(3), 312–320.

Conesa, J. C. and D. Krueger (1999, October). Social security reform with heterogeneous agents. *Review of Economic Dynamics 2*(4), 757–795.

Constantinides, G. M. and D. Duffie (1996). Asset pricing with heterogeneous consumers. *Journal of Political Economy 104*(2), 219–240.

Croce, M. M. (2010, May). Long-run productivity risk: A new hope for production-based asset pricing? *SSRN eLibrary*.

Deaton, A. (1991, September). Saving and liquidity constraints. *Econometrica 59*(5), 1221–1248.

Diamond, P. A. (1965). National debt in a neoclassical growth model. *American Economic Review 55*(5), 1126–1150.

Diamond, P. A. (1977). A framework for social security analysis. *Journal of Public Economics 8*(3), 275–298.

Fehr, H. and C. Habermann (2008). Risk Sharing and Efficiency Implications of Progressive Pension Arrangements. *Scandinavian Journal of Economics 110*(2), 419–443.

Fernández-Villaverde, J. and D. Krueger (2011). Consumption and saving over the life cycle: How important are consumer durables? *Macroeconomic Dynamics 15*(05), 725–770.

Gomes, F. J. and A. Michaelides (2008). Asset pricing with limited risk sharing and heterogeneous agents. *The Review of Financial Studies 21*(1), 415–448.

Gomes, F. J., A. Michaelides, and V. Polkovnichenko (2008, December). Fiscal policy in an incomplete markets economy. Working Paper, London Business School and CEPR.

Goodman, L. A. (1960). On the Exact Variance of Products. *Journal of the American Statistical Association 55*(292), 708–713.

Gordon, R. H. and H. R. Varian (1988). Intergenerational Risk Sharing. *Journal of Public Economics 37(2)*, 185–202.

Harenberg, D. and A. Ludwig (2013). The Welfare Effects of Social Security in a Stylized Overlapping Generations Model with Aggregate and Idiosyncratic Risk.

Harenberg, D., A. Ludwig, and J. Maus (2013). Is social security a good thing when the correlation of wages and returns is low?

Huggett, M. (2011). Sources of lifetime inequality. *American Economic Review 101*(7), 2923–54.

Huggett, M. and J. C. Parra (2010). How well does the u.s. social insurance system provide social insurance? *Journal of Political Economy 118*(1), 76–112.

Huggett, M., G. Ventrua, and A. Yaron (2011). Sources of Lifetime Inequality.

Huggett, M. and G. Ventura (1999). On the Distributional Effects of Social Security Reform. *Review of Economic Dynamics 2*, 498–531.

Imrohoroğlu, A., S. Imrohoroğlu, and D. H. Joines (1995, June). A life cycle analysis of social security. *Economic Theory 6*(1), 83–114.

Imrohoroğlu, A., S. Imrohoroğlu, and D. H. Joines (1998). The effect of tax-favored retirement accounts on capital accumulation. *American Economic Review 88*(4), 749–768.

Krueger, D. and F. Kubler (2006, June). Pareto-improving social security reform when financial markets are incomplete!? *American Economic Review 96*(3), 737–755.

Krusell, P. and J. A. A. Smith (1997). Income and wealth heterogeneity, portfolio choice, and equilibrium asset returns. *Macroeconomic Dynamics 1*(02), 387–422.

Krusell, P. and J. A. A. Smith (1998, October). Income and wealth heterogeneity in the macroeconomy. *Journal of Political Economy 106*(5), 867–896.

Ludwig, A. (2007). The gauss–seidel–quasi-newton method: A hybrid algorithm for solving dynamic economic models. *Journal of Economic Dynamics and Control 31*(5), 1610–1632.

Ludwig, A. and M. Reiter (2010). Sharing Demographic Risk - Who is Afraid of the Baby Bust? *American Economic Journal: Economic Policy 2(4)*, 83–118.

Mankiw, N. G. (1986). The Equity Premium and the Concentration of Aggregate Shocks. *Journal of Financial Economics 17*, 211–219.

Matsen, E. and O. Thogersen (2004). Designing Social Security - a Portfolio Choice Approach. *European Economic Review 48*, 883–904.

Merton, R. C. (1983). On the Role of Social Security as a Means for Efficient Risk Sharing in an Economy Where Human Capital Is Not Tradable. In Z. Bodie and J. B. Shoven (Eds.), *Financial aspects of the United States pension system*, pp. 325–58. Chicago: University of Chicago Press.

Nishiyama, S. and K. Smetters (2007). Does Social Security Privatization Produce Efficiency Gains? *Quarterly Journal of Economics 122*, 1677–1719.

Olovsson, C. (2010). Quantifying the Risk-Sharing Welfare Gains of Social Security. *Journal of Monetary Economics 57*, 364–375.

Shiller, R. J. (1999). Social Security and Institutions for Intergenerational, Intragenerational, and International Risk-Sharing. *Carnegie- Rochester Conference Series on Public Policy 50(99)*, 165–204.

Siegel, J. J. (2002). *Stocks for the Long Run : The Definitive Guide to Financial Market Returns and Long-Term Investment Strategies*. New York: McGraw-Hill.

Storesletten, K., C. Telmer, and A. Yaron (2004a). Cyclical Dynamics in Idiosyncratic Labor Market Risk. *Journal of Political Economy 112*, 695–717.

Storesletten, K., C. I. Telmer, and A. Yaron (1999). The risk-sharing implications of alternative social security arrangements. *Carnegie-Rochester Conference Series on Public Policy 50*, 213–259.

Storesletten, K., C. I. Telmer, and A. Yaron (2004b, June). Cyclical dynamics in idiosyncratic labor market risk. *Journal of Political Economy 112*(3), 695–717.

Storesletten, K., C. I. Telmer, and A. Yaron (2007, October). Asset pricing with idiosyncratic risk and overlapping generations. *Review of Economic Dynamics 10*(4), 519–548.

Weil, P. (1989). The Equity Premium Puzzle and the Risk-Free Rate Puzzle. *Journal of Monetary Economics 24*, 401–421.