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Three Solutions to the Pricing Kernel Puzzle*

Thorsten Hens†, Christian Reichlin‡

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Abstract

The pricing kernel puzzle is the observation that the pricing kernel might be increasing in some range of the market returns. This paper analyzes the pricing kernel in a financial market equilibrium. If markets are complete and investors are risk-averse and have common and true beliefs, the pricing kernel is a decreasing function of aggregate resources. If at least one of these assumptions is violated, the pricing kernel is not necessarily decreasing. Thus, incomplete markets, risk-seeking behaviour and incorrect beliefs can induce increasing parts in the pricing kernel and can be seen as potential solutions for the pricing kernel puzzle. We construct examples to illustrate the three explanations. We verify the robustness of the explanations under aggregation and compare the phenomena with the findings in the empirical literature. The results are used to reveal strengths and weaknesses of the three solutions. Risk-seeking behaviour is a fragile explanation that can only work in a model with atomic state space. Biased beliefs are robust under aggregation and consistent with the empirical findings. In incomplete markets, it is easy to find a pricing kernel with increasing parts. In order to get situations where all pricing kernels have increasing parts, we need extreme assumptions on the wealth distribution.

Keywords: Pricing kernel puzzle; Financial market equilibrium; Risk-seeking behaviour; Biased beliefs; Incomplete markets
JEL classification: D53,G12

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1 Introduction

A puzzle is a robust empirical observation that seems to contradict standard theory. In particular, the field of Finance is famous for its many puzzles (see, e.g., Kritzman (2002)). Most famous examples of puzzles in Finance include the “equity premium puzzle”, “the excess volatility puzzle” and the “pricing kernel puzzle”. The equity premium puzzle is the observation that the risk aversion estimated from market data is very high relative to the risk aversion observed on individual data, e.g., that obtained from laboratory experiments. The excess volatility puzzle is the observation that the volatility observed in stock market data is very high as compared to the volatility that results in standard asset pricing models. The pricing kernel is the quotient of the Arrow security prices and the objective probability measure. It is the “characteristic function” of an asset pricing model, also called the likelihood ratio process, that summarizes all relevant asset pricing information. In an economy with complete markets and risk-averse investors having common and correct expectations (i.e. all subjective expectations coincide with the objective probability measure) the pricing kernel is a decreasing function of aggregate resources, which in Finance are usually proxied by the returns of the market portfolio. The pricing kernel puzzle is the observation that however the pricing kernel might be *increasing* in some range of the market returns.
From a financial equilibrium point of view, most puzzles in Finance result from a too simplistic choice of the so-called “standard model”. Aggregate utility functions are chosen to be of the same type as individual utility functions. This is usually justified by Pareto-efficiency according to which the heterogeneity of agents can be aggregated into one representative agent (Negishi, 1960; Constantinides, 1982). Without restricting the degree of heterogeneity it is well known that a financial equilibrium can generate any arbitrage free price paths (Mas-Colell, 1977), which holds true for complete and also incomplete markets. Hence allowing for any degree of heterogeneity neither the equity premium nor the excess volatility puzzle are really puzzling, independent of whether the market is complete or incomplete. This is however different for the pricing kernel puzzle since with complete markets for no degree of heterogeneity this puzzle can be explained with risk-averse agents having correct beliefs. Hence the well-known “anything goes” result of Sonnenschein, Debreu and Mantel as it was called by Mas-Colell et al. (1995) has no bite for the pricing kernel puzzle!

In this paper, we explain the pricing kernel puzzle in financial markets with non-concave utilities, incorrect beliefs and incomplete markets. Any of these reasons can induce increasing parts in the pricing kernel. The paper also shows that a combination of these independent reasons leads to very simple and realistic economies explaining the pricing kernel puzzle. Simple exam-

ples for the three reasons are constructed to illustrate the three phenomena. We verify the robustness of the explanations under aggregation and compare the phenomena with the findings in the empirical literature. The results are used to deduce strengths and weaknesses of the explanations.
Risk-seeking behaviour is a model-dependent and fragile explanation and there is no theoretical justification for risk-seeking behaviour on the aggregate level. In order to be able to analyze the effect of incorrect beliefs, we need to separate different sources for incorrect beliefs as heterogeneity of beliefs, misestimation and distorted beliefs. While heterogeneous beliefs and misestimation lead to conceptual problems in measurement, distorted beliefs refer to a bias in the decision problem of the agents. Heterogeneous beliefs aggregate in some way and the pricing kernel with respect to the aggregate belief is decreasing. Distorted beliefs are robust under aggregation and consistent with the empirical findings. In incomplete markets, it is easy to find a pricing kernel with increasing parts. In order to get situations where all pricing kernels have increasing parts, we need extreme assumptions on the wealth distribution.

In the literature, the pricing kernel is mainly analyzed from the econometric viewpoint. Researchers have taken great interest in estimating the pricing kernel. One often-used approach relies on a model of a representative agent in which the pricing kernel is a parametric function of the aggregate endowment. Stock market data is then used to estimate the parameters. Two of numerous examples are Brown and Gibbons (1985) and Dittmar (2002). Hansen and Singleton (1983) additionally use consumption data for the estimation. Another approach is based on the no-arbitrage principle. While the techniques of this approach have become more and more sophisticated, the basic approach has remained the same. Along the lines of Breeden and Litzenberger (1978), option data is used to estimate the risk-neutral distribution. Other methods are used to estimate the historical distribution. Some examples are Jackwerth and Rubinstein (1996); Aït-Sahalia and Lo (1998); Jackwerth (2000); Aït-Sahalia and Lo (2000); Brown and Jackwerth (2001); Rosenberg and Engle (2002); Yatchew and Härdle (2006); Barone-Adesi et al. (2008); Barone-Adesi and Dall'O (2009). The main observation in that part of the literature is that the decreasing relation between aggregate resources and the pricing kernel may be violated. There is an interval usually in the area of zero return where the pricing kernel is increasing.
In the papers mentioned above, one finds many hypotheses which are evoked to solve the pricing kernel puzzle. One hypothesis is mistakes in the estimation, such as a faulty estimation procedure for the risk-neutral or the historical distribution or noisy option data. Jackwerth (2000) studies such explanations. Simulation studies in Chabi-Yo et al. (2007) suggest to explain the puzzle by regime switches of the prices for the underlying of the financial markets. Behavioral explanations have also been put forward. Many em-

pirical studies state that the increasing pricing kernel is strong evidence for risk-seeking behaviour of the representative agent. Shefrin (2005) explains the puzzle using a model with heterogeneous beliefs. Ziegler (2007) tests this explanation and concludes that the degree of heterogeneity required to explain the puzzle is implausibly large. More recently, distorted beliefs are used in Polkovnichenko and Zhao (2009). Härdle et al. (2009) transferred the regime switching explanation to a microeconomic perspective and suggest state dependent utilities as an explanation for the puzzle. Our paper gives a simple unifying framework of a financial market in which all of these hypotheses can be analyzed and compared. The general idea of our paper is that trade in financial markets leads to phenomena different to those expected from individual portfolio optimization.
The paper is organized as follows. In Section 2, we introduce the model and we define our notation of a financial market equilibrium. Section 3 considers the case of risk-averse agents having true and common beliefs in a complete market. Section 4 is devoted to the study of the case of partially risk-seeking agents. Section 5 provides a detailed exposition of the case that risk-averse agents have incorrect beliefs. In Section 6, we look more closely at the problem in incomplete markets. Section 7 presents a simple example which combines incomplete markets and heterogeneous beliefs. Finally, Section 8 concludes the main results. In an effort to keep clear the main lines of the argument, some of the drier mathematical calculations are placed in appendices. For standard results in financial theory, the corresponding results in Magill and Quinzii (1996) are cited as one possible reference.

2 Setup

We consider a two-period exchange economy. Let $\Omega = \{1, \ldots, S\}$, $S < \infty$ denote the states of nature in the second period. The set $\mathcal{F} = 2^{\Omega}$ is the power algebra on Ω, i.e., the set of all possible events arising from Ω. Uncertainty is modeled by the probability space $(\Omega, \mathcal{F}, P)$, where the probability measure P on Ω satisfies $p_s = P[\{s\}] > 0$ for all $s = 1, \ldots, S$, i.e., every state of the world has strictly positive probability to occur.
There are $K+1$ assets, whose payoffs at date $t = 1$ are described by $A_k \in \mathbb{R}^S$. The asset 0 is the risk-free asset with payoff $A_0 = 1$. The price of the k-th asset at date $t = 0$ is denoted by q_k. The risk-free asset supply is unlimited and the price q_0 is exogenously given by 1. This assumption does not restrict the generality of the model as we always may choose the bond as numeraire. In other words, the payoffs are already discounted. The prices of the other assets are endogenously derived by demand and supply. The market subspace $\mathcal{X}$ is the span of $(A_k)_{k=0,1,\ldots,K}$. Without loss of generality, we assume that no asset is redundant, i.e., $\dim(\mathcal{X}) = K + 1$, where obviously $K + 1 \leq S$ holds. The market is called complete if $K + 1 = S$ holds.

We consider a finite set $\mathcal{I}$ of investors. Agent i has a stochastic income $W^i \in \mathbb{R}_+^S$ at date 1. The variable $\theta^i = (\theta_0^i, \ldots, \theta_K^i) \in \mathbb{R}^{K+1}$ denotes the i-th agent's portfolio giving the number of units of each of the $K+1$ securities purchased (if $\theta_k^i > 0$) or sold (if $\theta_k^i < 0$) by agent i. Buying and selling these $K+1$ securities is the only trading opportunity available to agent i. Thus, given the available securities, investor i can attain any payoff $X = W^i + \sum_{k=0}^K A_k \theta_k^i$, where θ^i satisfies the budget restriction $\sum_{k=0}^K q_k \theta_k^i \leq 0$. Moreover, we assume that the resulting income must be positive in all states of nature, i.e., $X \geq 0$. The subset of payoffs in $\mathcal{X}$ that are positive and budget feasible for investor i is denoted by $B^i(q)$, i.e.,

$$B^i(q) := \left\{ X \in \mathbb{R}_+^S \,\middle|\, X = W^i + \sum_{k=0}^K A_k \theta_k^i \text{ for } \theta^i \in \mathbb{R}^{K+1} \text{ s.t. } \sum_{k=0}^K q_k \theta_k^i \leq 0 \right\}.$$

The preferences of agent i are described by an increasing functional $V^i : \mathcal{X} \to \mathbb{R}$. This functional summarizes the utility function as well as the beliefs of the agents. We will explicitly define functionals in the next sections. In order to optimize the preference functional, agents may want to buy and sell assets. A new allocation $(X^i)_{i \in \mathcal{I}}$ is called feasible if the resulting total demand matches the overall supply. Formally, this means that the market-clearing condition

$$\sum_{i \in \mathcal{I}} \theta^i = 0$$

has to be satisfied. Note that the market-clearing conditions for the financial contracts imply that the allocation $(X_i)_{i \in \mathcal{I}}$ satisfies

$$\sum_{i \in \mathcal{I}} X^i = \sum_{i \in \mathcal{I}} W^i.$$

In an ideal situation, prices of the assets are derived in such a way that the requested profiles X^i form a feasible allocation.

Definition. *A price vector $\hat{q} = (1, \hat{q}_1, \ldots, \hat{q}_K)$ together with a feasible allocation $(\hat{X}^i)_{i \in \mathcal{I}}$ is called a financial market equilibrium if each $\hat{X}^i$ maximizes the functional V^i subject to $B^i(\hat{q})$.*

Since the preference functional is strictly increasing, the agents would exploit arbitrage opportunities in the sense of a sure gain without any risk. This means that if there were such an opportunity, every agent would rush to exploit it and so competition will make it disappear very quickly. Thus, we conclude that the condition

$$\left\{ X \in \mathbb{R}_+^S \,\middle|\, X = \sum_{k=0}^K A_k \theta_k^i \text{ for } \theta^i \in \mathbb{R}^{K+1} \text{ s.t. } \sum_{k=0}^K q_k \theta_k^i \leq 0 \right\} = \{0\}$$

is satisfied in equilibrium. This implies (Magill and Quinzii, 1996, Theorem 9.3) the existence of strictly positive Arrow security prices $\pi = (\pi_1, \dots, \pi_S)$ summing to 1 such that $q_k = \pi A_k$ holds for all assets k. Each set of such Arrow security prices then defines a pricing kernel

$$\frac{\pi}{p} := \left(\frac{\pi_1}{p_1}, \dots, \frac{\pi_S}{p_S}\right).$$

Note that the pricing kernel is not unique if the market is incomplete (Magill and Quinzii, 1996, Theorem 10.6).

3 The Pricing Kernel Puzzle

In the introduction, we defined a puzzle as an observation that seems to contradict the standard theory. In this section, we first explain what is meant by standard theory and we will give a short overview over the main findings of the empirical literature.

In the main part of the finance literature, agents are assumed to be risk-averse and to have common and true beliefs. Formally, the preference functional V^i of agent i is then given by

$$V^i(X) := E\left[U^i(X)\right] = \sum_{s=1}^{S} p_s U^i(X_s)$$

for a strictly increasing, strictly concave utility function $U^i : \mathbb{R}_+ \to \mathbb{R}$ satisfying the Inada-conditions

$$\begin{aligned} U^{i\prime}(0) &:= \lim_{x \searrow 0} U^{i\prime}(x) = +\infty, \\ U^{i\prime}(\infty) &:= \lim_{x \to \infty} U^{i\prime}(x) = 0. \end{aligned}$$

Markets are often assumed to be complete. Under these assumptions, it follows that there is a decreasing relation between the pricing kernel and aggregate resources.

Lemma 1. *Consider a financial market satisfying* $\dim(\mathcal{X}) = S$ *and let the preference functionals* V^i *be given as above. If* $(\hat{q}, (X^i)_{i\in\mathcal{I}})$ *is a financial market equilibrium with pricing kernel* $\frac{\pi}{p}$*, then there exists a strictly decreasing function* $f : \mathbb{R}_+ \to \mathbb{R}_+$*, such that*

$$\frac{\pi_s}{p_s} = f(W_s), \quad s = 1, \dots, S.$$

A formal proof is given in Magill and Quinzii (1996), Theorem 16.7. Intuitively, every agent (or even simpler, the representative agent) forms his portfolio according to the first-order conditions, i.e., the requested profile has the form

$$\hat{X}^i_s = \left(U^{i\prime}\right)^{-1}\left(\lambda^i \frac{\pi_s}{p_s}\right)$$

for a suitable Lagrange parameter λ^i. Because of the decreasing marginal rate of substitution, this profile is a decreasing function of the pricing kernel. The same holds true for the sum of all profiles of the agents. Due to the market-clearing condition, this sum is equal to the aggregate resources. This implies that aggregate resources are a decreasing function of the pricing kernel.

Remark. The assumptions of Lemma 1 can be relaxed. It is enough to assume that the utility functions U^i are increasing and concave (i.e., not necessarily strictly concave and not necessarily satisfying the Inada-conditions). Indeed, Theorem 1 of Dybvig (1988) and its generalisation in Appendix A of Dybvig (1988) show that the allocation $\hat{X}^i$ of agent i and the pricing kernel are anti-comonotonic[1]. Hence, this also holds for the sum over all agents. Using the market-clearing condition, it follows that the sum $W = \sum_{i\in\mathcal{I}} \hat{X}^i$ and the pricing kernel are anti-comonotonic. $\diamond$

Loss aversion is the observation that people strongly prefer avoiding losses to acquiring gains. Loss aversion is the most robust aspect of prospect theory and it was first convincingly demonstrated in Kahneman and Tversky (1979). The utility of a loss-averse person is often modeled by a piecewise linear function. It is linear in the loss domain and in the gain domain; but it is steeper in the loss part than in the gain part. Such a utility is concave and thus leads to a decreasing pricing kernel.

Example 1. If we restrict ourselves to mean-variance type preferences, we end up in the CAPM which is the traditional example in equilibrium theory. There, the pricing kernel is an affine decreasing function of the aggregate resources (Magill and Quinzii, 1996, Theorem 17.3).

Lemma 1 and Example 1 presented a set of assumptions which implies a decreasing relation between the pricing kernel and aggregate resources. However, there is strong empirical evidence that this decreasing relation may be violated. This observation was made by several authors using different methods and different data sets (Jackwerth, 2000; Aït-Sahalia and Lo, 2000; Brown and Jackwerth, 2001; Rosenberg and Engle, 2002; Yatchew and Härdle, 2006; Barone-Adesi and Dall'O, 2009). Furthermore, the estimated form

[1] Comonotonicity of two random variables intuitively means that their realizations have the same rank order. In our setup, two random variables X^1 and X^2 are called comonotonic if $\left(X^1_s - X^1_{s'}\right)\left(X^2_s - X^2_{s'}\right) \geq 0$ for all $s, s' \in \{1, \ldots, S\}$. Random variables X^1 and X^2 are called anti-comonotonic if X^1 and $-X^2$ are comonotonic. See Föllmer and Schied (2004) for a general definition (Definition 4.76) and equivalent formulations (Lemma 4.83).

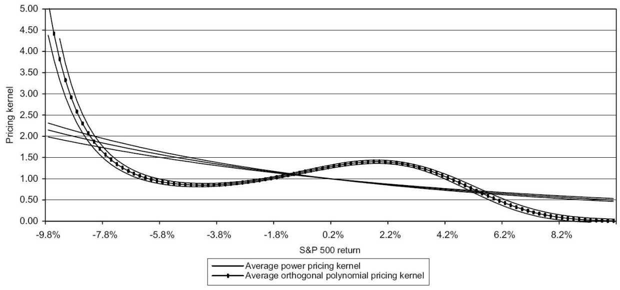


Figure 1: Rosenberg and Engle (2002), page 21

of the pricing kernel is stable as well. Linear functions that fit the pricing kernel well, are decreasing. Using more flexible estimations, there is an interval usually in the area of zero returns where the pricing kernel is increasing. A typical form is presented in Figure 1. Note however, that a bumpy pricing kernel is mainly derived using option data. Using market data, it seems to be more difficult to show (significantly) the presence of an increasing part of the pricing kernel (Schweri (2010)). Thus we find it important to really understand the reasons for the shape of the pricing kernel. In the following three sections, we alternately skip one of the three main assumptions complete markets, risk-aversion and correct beliefs and try to understand how the skipped assumption influences the pricing kernel.

4 S-shaped utility

In this section, we consider partially risk-seeking agents in a complete market ($\dim(\mathcal{X}) = S$). More precisely, the agents have common and true beliefs, but they are not necessarily risk-averse. While risk aversion is a standard assumption in Finance, there is considerable empirical evidence that agents might show risk aversion for some ranges of returns and risk-seeking behaviour for others (typical examples can be found in Kahneman and Tversky (1979)). Formally, the preference functional is described by

$$V^i(X) = E[U^i(X)],$$

where $U^i : \mathbb{R}_+ \to \mathbb{R}$ is strictly increasing. Hence, the main difference to the situation of Lemma 1 is that U^i is not necessarily concave. In the literature, the most prominent examples of non-concave utilities are the concave-convex-concave utility function suggested by Friedman and Savage (1948) and the convex-concave utility arising in Prospect theory (Kahneman and Tversky, 1979; Tversky and Kahneman, 1992).

In several papers on the pricing kernel puzzle, a representative agent economy is considered. In such an economy, the prices are chosen in such a way that the aggregate resource W is an optimal allocation for the representative agent with initial endowment W. Optimality implies that the first-order condition is satisfied, i.e., the equality $U'(W) = \lambda \frac{\pi}{p}$ has to hold. Following this line of arguments, the pricing kernel represents the shape of the utility function of the representative agent. Considering Figure 1, this leads to a concave-convex-concave utility function with the upper reference point at 2.2%. The form of the representative utility function coincides with the form suggested by Friedman and Savage. Moreover, the reference point is in the area between zero return and the inflation rate. Such a reference point is consistent with Prospect theory. The next example shows how a state in the convex area can lead to an increasing pricing kernel.

Example 2. We consider an economy with two states, two assets and a single (representative) agent. The underlying probabilities are defined by $p_1 = \frac{2}{3}$ and $p_2 = \frac{1}{3}$. The payoff matrix of the assets is given by

$$\mathbf{A} = \begin{pmatrix} 1 & \frac{2}{3} \\ 1 & 2 \end{pmatrix}.$$

The utility function of the agent is given by

$$U(x) = \begin{cases} (x-1)^{\frac{1}{3}}, & x \geq 1, \\ -(1-x)^{\frac{2}{3}}, & x < 1. \end{cases}$$

We can interpret 1 as his reference point. On the interval $(0, 1)$, the agent is risk-seeking and on the interval $(1, \infty)$, the agent is risk-averse. It is shown in Appendix A that $\hat{X} = (\frac{10}{9}, \frac{2}{3})$ is the optimal allocation of the agent for the prices $\hat{q} = (1, 1)$. Thus, endowing the agent with $W = (\frac{10}{9}, \frac{2}{3})$, we can interpret this as an equilibrium with a single (representative) investor. In order to analyze the pricing kernel, we first determine the Arrow security prices defined by the equation $1 = \pi A$. It follows that $\pi_1 = \frac{3}{4}$ and $\pi_2 = \frac{1}{4}$. Hence, the according unique pricing kernel is given by $\frac{\pi}{p} = (\frac{9}{8}, \frac{3}{4})$. We conclude that there is no decreasing relation between the pricing kernel and portfolio of the single investor.
For the intuition of this example, it is important to note that the probabilities of the states are not equal. State 1 has a higher probability of happening. Thus, the risk-seeking behaviour inveigles agent 1 to invest more in state 1 even though the pricing kernel of state 1 is higher.

The idea to explain the pricing kernel puzzle with non-concave utilities is, as Example 2 has shown in principle possible, however it is problematic for the following reasons. If the utilities are partially convex, the first-order condition is necessary for optimality, but it is not sufficient anymore. One can easily construct examples where several other non-optimal candidates satisfy the first-order condition. Thus, it has to be checked, as we did in Example 2, whether the allocation W is actually optimal for the utility derived via $U'(W) = \lambda \frac{\pi}{p}$. In order to verify the optimality, we use the following result, which is proved in Appendix B.

Lemma 2. *Let U be an increasing, smooth utility function with a convex area. Let C denote the interior of the convex area of the utility U and let X^* be the optimal allocation for the pricing kernel $\frac{\pi}{p}$ and initial endowment W. Then, there exists at most one state $s \in \{1, \ldots, S\}$ with $X_s^* \in C$.*

Let us give the intuition for this result: utilities with convex areas induce risk-seeking behaviour because one wants to end up on the concave hull. One branch of literature (Bailey et al., 1980; Hartley and Farrell, 2002; Levy and Levy, 2002) uses this idea to argue that convex areas do not influence the utility maximization problem (i.e., $P[X^* \in C] = 0$). This argument uses the assumption that one can split up every payoff. Formally, the crucial assumption is that the underlying probability space is atomless. This assumption is not satisfied in our setup. On an atomic probability space, not all bets are possible. Hence, it is not possible to design any payoff distribution. However, the above lemma shows that it is optimal to allocate in such a way that at most one state lies in the convex area. In the case that there are a lot of states, the influence of the single state becomes small.

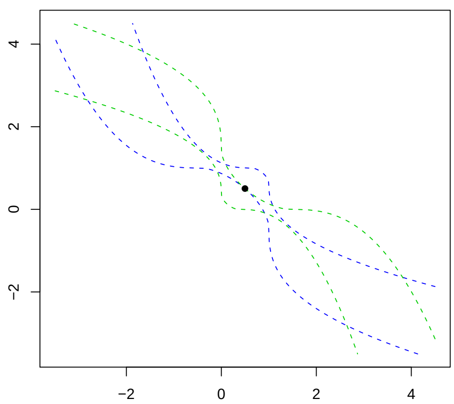
Let us relate this result to Figure 1. By way of contradiction, we assume that the allocation W is the optimal allocation of a representative agent for the given prices. The first-order condition tells us that the utility of the representative agent is concave-convex-concave. However, we see that the interval between -3.5% and 2% lies in the convex area, i.e., $P[W \in C] > 0$ and there is definitely more than one state taking values in the convex area because most of the return observations lie in this interval! This is a contradiction to Lemma 2. We conclude that the allocation W cannot be the optimal allocation for the representative agent with the utility derived via the first-order condition $U'(W) = \lambda \frac{\pi}{p}$.

Even if there is a state in the convex domain, it is not clear whether the pricing kernel does have increasing parts. Indeed, Theorem 1 of Dybvig (1988) shows that the optimal allocation of agents with increasing, not necessarily concave utility functions is a decreasing function of the Arrow security prices if the states have equal probability.

A second problem of partially convex utility functions lies in the aggregation. It is well known that risk-averse agents aggregate to a representative risk-averse agent, i.e., the behaviour of a class of risk-averse agents transfer to the same behaviour of the representative agent. The same does not hold true for (partially) risk-seeking agents. The representative agent of two identical agents with a utility function according to Prospect theory does not necessarily behave as an agent with such a utility function. As an illustration, we consider two agents with initial endowment $W^1 = W^2 = (\frac{1}{2}, \frac{1}{2})$. The utility function of the agents is given by

$$U(x) = \begin{cases} (x-1)^{\frac{1}{3}}, & x \geq 1, \\ -(1-x)^{\frac{2}{3}}, & x < 1. \end{cases}$$

We illustrate this situation in a Edgeworth box.



Agent 1's quantities are measured in the usual way, with the southwest corner as the origin. The blue lines represent some indifference curves of agent 1. They contain concave areas. Agent 2's quantities are measured using the northeast corner as the origin. The green lines represent some indifference curves of agent 2. Looking from the northeast corner, the curves have the same form. The initial endowment is represented by the black circle. The picture suggests that there are at least two equilibria. The aggregate resources are $(1, 1)$. In order to represent the situation, the aggregate resources should be the optimal allocation for the given prices and the given representative utility function. The first guess for the representative utility function is the utility U, because both agents are of this type. But, considering such a utility function U, the utility maximization problem in Appendix A shows that there are no prices such that staying at the reference point is optimal.

We conclude that risk-seeking agents can be seen as a possible argument for increasing areas in the pricing kernel. However, it is model-dependent and fragile and it is dangerous to transfer the common arguments for Prospect theory on the individual level to the aggregate utility function.

5 Incorrect beliefs

Let us now extend the discussion to situations where agents do not necessarily have common and true beliefs. In such a setting, the risky probabilities used in the pricing kernel do not coincide with the weights used by the agents for evaluation of the payoffs. In order to be able to analyze the shape of the pricing kernel and the robustness under aggregation, we need to separate different sources for incorrect beliefs. In the literature on the pricing kernel puzzle, heterogeneous beliefs and misestimation are considered as possible explanations. These explanations amount to a conceptual problem in measurement: by which p do we define the pricing kernel? Considering agents with heterogeneous beliefs, this question is of fundamental importance; one has to carefully deal with aggregation of beliefs. On the other hand, analyzing the estimation procedure leads to practical problems arising in the measurement of the pricing kernel. We suggest to additionally consider distortions, which amounts to a bias in decision making. We first consider the utility maximization problem for general incorrect beliefs and we then analyze the three phenomena independently.

In order to formally describe incorrect beliefs, we assume that the preferences of agent $i \in \mathcal{I}$ are described by the functional

$$V^i(X) := E_{P^i}\left[U^i(X)\right] = \sum_{s=1}^{S} p_s^i U^i(X_s),$$

where P^i is a set function on $(\Omega, \mathcal{F})$. The set function P^i represents the subjective expectations of agent i about the future. For simplicity, we consider again the case of strictly concave utility functions satisfying the Inada conditions. In equilibrium, the agents maximize their utility functional V^i over the budget set $B^i(\hat{q})$. In particular, the payoffs are evaluated with respect to their own belief P^i. Formally, the allocation $\hat{X}^i$ of agent i solves the maximization problem

$$\text{maximize } E_{P^i}\left[U^i(X)\right] \text{ subject to } X \in B^i(\hat{q}).$$

Using the relation $\hat{q} = \pi A$, the constraint $\sum_{k=0}^{K} q_k \theta_k^i \le 0$ can be rewritten (for details see Magill and Quinzii (1996), Page 83f) as $\sum_{s=1}^{S} \pi_s X_s \le \sum_{s=1}^{S} \pi_s W_s^i$. The modification allows the Lagrange method to be used and we conclude that the requested profile of agent i has the form

$$\hat{X}_s^i = \left(U^{i\prime}\right)^{-1}\left(\lambda^i \frac{\pi_s}{p_s^i}\right)$$

for a suitable Lagrange parameter λ^i. Note that the "true" real-world probability does not appear in the profile. The agent invests proportional to the ratio of the price of the corresponding Arrow security and the subjective probability P^i of its success. Hence, the requested profile $\hat{X}^i$ of agent i is a decreasing function of the pricing kernel $\frac{\pi}{p^i}$ with respect to his subjective probability measure. However, it is not necessarily a decreasing function of the pricing kernel $\frac{\pi}{p}$ with respect to the "true" probability measure. Thus, in a model with a single (representative) agent, a bump can be viewed as a difference between his subjective probability measure and the "true" measure.

Let us now consider distortions as a special case for biased beliefs. Kahneman and Tversky (1979) show that agents tend to overweight extreme events. The simplest way to incorporate such a behaviour is to distort the given true probabilities of the states with an increasing, concave-convex function $T : [0, 1] \to [0, 1]$. The agent then evaluates the payoff with respect to the distorted probability $T(p_.)$, i.e., the preference functional is given by

$$V(X) = \sum_{s=1}^{S} U(X_s)T(p_s).$$

Using the arguments above, we find that the requested profile is

$$\hat{X}^i_s = (U')^{-1} \left(\lambda \frac{\pi_s}{T(p_s)} \right).$$

It is a decreasing function of $\frac{\pi_.}{T(p_.)}$, but it is not necessarily decreasing in $\frac{\pi_.}{p_.}$. This holds true for every agent and we conclude that the pricing kernel with respect to distorted probabilities is a decreasing function of the aggregate resources. However, we plot $\frac{\pi}{p}$ as a function of W_s. For a state with high probability (e.g., returns around zero in the case of S&P 500), $T(p_s)$ is relatively underestimated. Hence, $\frac{\pi}{T(p_s)}$ is relatively higher than $\frac{\pi}{p_s}$. For a state with low probability (e.g., extreme returns in the case of S&P 500), $T(p_s)$ is relatively overestimated and $\frac{\pi}{T(p)}$ is relatively lower than $\frac{\pi}{p}$. Following this argumentation, the pricing kernel $\frac{\pi}{p}$ has an increasing interval in the area that occurs with high probability. Thus, this gives a simple and robust explanation of the pricing kernel puzzle.

In reality, different agents may have different views about the future, i.e., the beliefs P^i may differ. As argued above, the profile requested by agent i is a decreasing function of $\frac{\pi}{p^i}$. On the first view, it seems to be difficult to make precise statements about the pricing kernel. However, there are aggregation results for particular situations. One example is the CAPM with heterogeneous beliefs about the means. In such a situation, the aggregate belief can explicitly be derived (Gerber and Hens, 2006, Proposition 2.1). It takes into account both the relative wealth and also the risk aversion of the agents. The wealthier and the less risk-averse agents determine the consensus belief more than the poor and more risk-averse agents. The pricing kernel with respect to the derived aggregate belief is then a linear decreasing function of the aggregate resources (Hens and Rieger, 2010, Section 4.4.1). Another prominent example can be found in (Shefrin, 2005, Theorem 14.1). It shows how agents with three sorts of heterogeneity (risk aversion, time discount factor and belief) aggregate into a single representative investor. The pricing kernel with respect to the belief of the representative agent is then a decreasing function of aggregate resources. In the case of arbitrary concave utility functions U^i and arbitrary beliefs P^i, it is possible (Calvet et al. (2001), Theorem 3.2 and Jouini and Napp (2006), Proposition 2.1) to define another common 'consensus' belief which, if held by all agents, would (after a possible reallocation of the initial endowments) generate the same equilibrium prices as in the actual heterogeneous world. The risk-neutral probabilities in the 'equivalent' equilibrium remain the same because they just depend on the prices. Moreover, all the requested profiles in the equivalent equilibrium are decreasing functions of the pricing kernel with respect to the common 'consensus' belief. This transfers to the sum of all profiles which corresponds to the aggregate resources. So, even in the most general case, there is a belief such that the pricing kernel with respect to this belief is a decreasing function of the aggregate resources. A difference between the "true" probability and this belief is necessary for a partially increasing pricing kernel. Thus, heterogeneity of beliefs does not give a good explanation of the pricing kernel puzzle.

The arguments before show that there is a belief such that the pricing kernel with respect to this belief is decreasing. In order to estimate the pricing kernel, the true probability has to be estimated somehow. Often, this is done using the historical distribution. However, this "past" probability can certainly differ from the representative beliefs $\tilde{P}$. Let us consider the case of a regime switch. The beliefs of the agent will adopt the new situation quite fast, but at the beginning this change is not reflected in the historical distribution. This implies a difference between the estimated "true" probability $\tilde{P}$ and the representative belief, and this difference destroy the decreasing relation between the pricing kernel and aggregate resources. After some time, the new beliefs are also reflected in the historical distribution, the estimated "true" belief P approximately represents the representative belief and the bump disappears. The illustration before indicates that the pricing kernel does not necessarily have a bump all the time. It may have an increasing part at some special points in time, but not necessarily all the time. Most interesting are bullish and bearish regimes. Empirically, it is well-documented that the volatility in returns in bearish-regimes is higher than in bullish-regimes. This so-called "Leverage effect" was first documented by Black (1976) and Christie (1982). See Bekaert and Wu (2000) for an overview. It follows that the probabilities of extreme/normal returns in bearish-regimes are higher/lower than those probabilities in bullish-regimes. The according

change in the pricing kernel leads to shapes similar to the one observed in the empirical literature. In particular, Detlefsen et al. (2007) estimate the pricing kernel in different regimes and their results are consistent with our model.
There are also attempts to estimate the pricing kernel with market data. Using this method, it is implicitly assumed that the pricing kernel (in particular, the risk-neutral and the historical distribution) is constant over a long time period. The time periods where the representative belief is not reflected in the historical distribution are relatively short and thus, they may not influence the average over the long run. This explains the observation that the estimated form of the pricing kernel is usually decreasing using market data.

6 Incomplete markets and heterogeneous background risk

Up to now, we restricted ourselves to the case of a complete market economy, i.e., $\dim(\mathcal{X}) = S$. In this section, we want to analyze the case $\dim(\mathcal{X}) < S$. For simplicity, we isolate this extension in the sense that agents are risk-averse and have common and true beliefs.
Let us again consider an equilibrium, i.e., there are prices $\hat{q}$ such that every agent maximizes the preference functional subject to his budget set and the market-clearing condition is satisfied. More technically, each agent solves a problem of the form

$$\max E[U^i(X)] \text{ over } X \in B^i(\hat{q}).$$

In the case of complete markets, the requested profile is a decreasing function of the pricing kernel. In incomplete markets, there are infinitely many risk-neutral measures π satisfying the equation $\hat{q} = \pi A$ and hence also infinitely many pricing kernels. The constraint $\sum_{k=0}^{K} q_k \theta_k^i \leq 0$ can be rewritten using the pricing kernels and writing down the first-order conditions of that problem, it turns out (Magill and Quinzii, 1996, Theorem 12.4) that the solution has the same form as in the complete market case for a particular pricing kernel. More precisely, for every agent i, there is a risk-neutral measure π^i such that the requested profile is of the form

$$\hat{X}^i = \left(U^{i\prime}\right)^{-1}\left(\lambda^i \frac{\pi^i}{p}\right).$$

This shows that every profile is a decreasing function of **a** pricing kernel. Hence, if there is a single representative agent, there exists some pricing kernel such that its profile is a decreasing function of that pricing kernel. With heterogeneous agents and incomplete markets, we can however give an example in which no pricing kernel is a decreasing function of aggregate resources. So, the pricing kernel does not reflect the value of an additional unit of wealth as in the complete case, but it also reflects the dependence between the states. We illustrate this phenomenon in the next example. In particular, we emphasize that a reduction to a single (representative) agent already excludes some interesting phenomena.

Example 3. We consider an economy with three states, two assets and two agents. The underlying probabilities are defined by $p_1 = p_2 = p_3 = \frac{1}{3}$. The payoff matrix of the assets is given by

$$\mathbf{A} = \begin{pmatrix} 1 & \frac{13}{15} \\ 1 & \frac{4}{3} \\ 1 & \frac{13}{15} \end{pmatrix}.$$

There are two agents. Both of them have utility $U^1(x) = U^2(x) = \ln(x)$ and they have common and true beliefs, i.e., they evaluate utilities according to the probabilities $p_1 = p_2 = p_3 = \frac{1}{3}$. The initial endowment is given by $W^1 = (\frac{76}{45}, \frac{19}{6}, \frac{212}{15})$ and $W^2 = (\frac{418}{15}, \frac{23}{6}, \frac{382}{285})$. It is shown in Appendix C that $\hat{q} = (1,1)$, $\hat{X}^1 = (\frac{14}{9}, \frac{7}{2}, 14)$ and $\hat{X}^2 = (28, \frac{7}{2}, \frac{28}{19})$ is a unique financial market equilibrium. In order to characterize the pricing kernels, we consider the risk-neutral probabilities. Every risk-neutral probability satisfies $\pi_1\frac{13}{15} + \pi_2\frac{4}{3} + \pi_3\frac{13}{15} = 1$ and it easily follows that all these probabilities can be written as a convex combination of the two extreme points $\left(0, \frac{2}{7}, \frac{5}{7}\right)$ and $\left(\frac{5}{7}, \frac{2}{7}, 0\right)$. We infer that $\pi_2 < \max(\pi_1, \pi_3)$ holds for every risk-neutral probability. Because of $p_1 = p_2 = p_3$, the same holds true for the pricing kernel, i.e., $\frac{\pi_2}{p_2} \leq \max(\frac{\pi_1}{p_1}, \frac{\pi_3}{p_3})$. The market portfolio $W = X^1 + X^2 = (\frac{266}{9}, 7, \frac{294}{19})$ has the lowest value in state 2. We infer that **no** pricing kernel is a decreasing function of the aggregate resources.

Let us interpret this example. According to the marginal rate of substitution, agent 1 would like to transfer wealth from state 3 to state 1 and agent 2 would like to transfer wealth from state 1 to state 3. However, there is no asset (combination) which does this job. If one increases wealth in state 1, the wealth in state 3 automatically increases and vice versa. It is only possible to transfer money from state 1 and state 3 to state 2 and vice versa. Both agents have high wealth in one state and relatively low wealth in the other two states. From a marginal-rate-of-substitution point of view, the low-wealth states are more important. Thus, both agents try to equalize the two states with the low value. More precisely, agent 1 would like to transfer money from state 2 to state 1 and agent 2 would like to transfer money of state 2 to state 3. This explains the low Arrow security price in state 2. But, the aggregate resources in state 2 are low. According to the marginal rate of substitution, an additional unit of wealth brings a high additional utility. We conclude that this information is, contrary to the case of complete markets, not contained in any pricing kernel. The pricing kernel reflects the dependence between the different states in the sense that both agents want to reduce their holding in state 2 relatively to the other states.

In reality, measurement of aggregate endowment is difficult. There have been approaches to use consumption data. However, there are many problems with imprecise measurements of these data. A popular alternative is to consider an equity index level as the projection of the aggregate endowment on the market subspace and to estimate the projected pricing kernel. Using this method, it is not clear whether or not increasing intervals that appear in the projected pricing kernel also appear in the true pricing kernel. In Example 3, we do not have this problem because we directly derived the true pricing kernel.

7 Combination of incomplete markets and heterogeneous beliefs

In the previous sections, we showed that one separate extension of the standard theory is sufficient to create an increasing interval in the pricing kernel. The next and last example considers agents with heterogeneous beliefs in incomplete markets. We derive conditions for the initial endowments leading to bumps in the pricing kernels.

Example 4. We consider an economy with three states, two assets and two agents. The underlying probabilities are defined by $p_1 = p_2 = p_3 = \frac{1}{3}$. The payoff matrix of the assets is given by

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

There are two agents. Both of them have utility $U^1(x) = U^2(x) = \ln(x)$, but they have subjective beliefs. The belief of agent 1 is given by $P^1 = (\frac{1}{2}, \frac{1}{2}, 0)$ and the belief of agent 2 is given by $P^2 = (\frac{1}{2}, 0, \frac{1}{2})$; agent 1 maximizes the consumption in state 1 and 2 and the other agent maximizes the consumption in 1 and 3. The initial endowments are given by $W^1 = \left(W_1^1, W_2^1, W_3^1\right)$ and $W^2 = \left(W_1^2, W_2^2, W_3^2\right)$ such that $W_1^1 + W_1^2 \leq W_2^1 + W_2^2 \leq W_3^1 + W_3^2$ holds. It is shown in Appendix D that

$$\hat{q} = \left(1, \frac{W_1^1 + W_1^2}{W_3^2 + W_2^1}\right)$$

is an equilibrium price. In order to determine the prices of the Arrow securities, we consider the relation $q = \pi A$, which simplifies to

$$\pi_1 = 1,$$
$$\pi_2 + \pi_3 = q_1.$$

If the initial endowment is chosen in such a way that $\frac{W_1^1+W_1^2}{W_3^2+W_2^1} > 2$, it holds that $\pi_1 \leq \max\{\pi_2, \pi_3\}$ and it follows that all pricing kernels have an increasing interval.

This example has a nice economical interpretation. There are two (classes of) agents. While agent 1 considers state 1 and 2 as possible realizations, agent 2 considers state 1 and 3 as possible realizations. In reality, we do not know which states are possible realizations. Hence, different views about possible realizations are very natural. And, as seen above, even in the most simple examples that incorporate such a behaviour, the pricing kernel is not necessarily decreasing. Moreover, due to the specific setting, the example turns out to be simple and solvable for general initial endowments. In this sense, it also gives an idea about the influence of the heterogeneous background risk on the pricing kernel. Thus, while each reason given above is in isolation sufficient to explain the pricing kernel puzzle, a combination of those reasons is most likely the case.

8 Conclusion

In an economy with complete markets and risk-averse investors having common and correct expectations the pricing kernel is a decreasing function of aggregate resources. We have shown that the assumptions complete markets, risk aversion of the investors and common and true beliefs are necessary assumptions for the statement. As soon as we relax one assumption, one can construct examples with a bumpy pricing kernel. In this sense, the pricing kernel puzzle results from a too-simplistic choice of the so-called "standard model".

The explanation that (partially) risk-seeking agents induce a bump is model-dependent. It only works on an atomic probability space. In a model with biased beliefs, the pricing kernel with respect to the representative belief is a decreasing function of aggregate resources. Bumps correspond to a difference between measured "true" probability and consensus probability. Distorted beliefs are robust under aggregation and consistent with the empirical findings. In incomplete markets, the pricing kernel also reflects information about the dependence between the different states.

To understand which of the three reasons we evoke to explain the puzzle is empirically most relevant, one would need to combine empirical studies on the pricing kernel with data on risk-seeking behaviour, probability weighting or incompleteness of markets. On the one hand, there are ways to measure the monotonicity of pricing kernels (Barone-Adesi and Dall'O (2009) and Golubev et al. (2008)); on the other hand, there are surveys on market structure, risk abilities and heterogeneity of beliefs in different countries (Wang et al. (2009)). As one possible approach, we therefore suggest to measure the monotonicity of the pricing kernel in different countries and compare the results with the data in the international survey. This analysis will give an idea which of the reasons seems to be most relevant.

A Example 2

The according risk-neutral probabilities for $\hat{q} = (1, 1)$ are $\pi_1 = \frac{3}{4}$ and $\pi_2 = \frac{1}{4}$. In order to show that the allocation $\hat{X}^1 = (\frac{10}{9}, \frac{2}{3})$ is optimal, we have to check that it solves

$$\max p_1 U(x_1) + p_2 U(x_2) \text{ subject to } \pi_1 x_1 + \pi_2 x_2 \leq \pi_1 \frac{10}{9} + \pi_2 \frac{2}{3} = 1.$$

Let us consider the cases $x_1 = x_2 = 1$, $x_1 > 1 > x_2$ and $x_1 < 1 < x_2$ independently. In the case $x_1 > 1 > x_2$, plugging in the constraint, differentiating the term with respect to x_1 and setting the resulting term equal to 0 give

$$\hat{x}_1 = 1 + \left(\frac{p_1}{2p_2}\right)^3 \left(\frac{\pi_2}{\pi_1}\right)^2,$$

$$\hat{x}_2 = 1 - \left(\frac{p_1}{2p_2}\right)^3 \left(\frac{\pi_2}{\pi_1}\right).$$

Plugging the candidate $\hat{x}_1$ into the second derivatives gives

$$-p_1 \frac{1}{3}\frac{2}{3}\left(\frac{p_1}{2p_2}\right)^{-5} \left(\frac{\pi_2}{\pi_1}\right)^{-\frac{10}{3}} + p_2 \frac{1}{3}\frac{2}{3}\left(\frac{\pi_2}{\pi_1}\right)^{-\frac{10}{3}} \left(\frac{p_1}{2p_2}\right)^{-4} < 0,$$

which shows that $(\hat{x}_1, \hat{x}_2)$ is indeed a local maximum. The expected utility is

$$\frac{p_1^2 \pi_2^{\frac{2}{3}}}{4 p_2 \pi_1^{\frac{2}{3}}}.$$

The same procedure for the case $x_2 > 1 > x_1$ shows that

$$\overset{*}{x}_1 = 1 - \left(\frac{\pi_1}{\pi_2}\right)\left(\frac{p_2}{2p_1}\right)^3,$$

$$\overset{*}{x}_2 = 1 + \left(\frac{\pi_1}{\pi_2}\right)^2 \left(\frac{p_2}{2p_1}\right)^3$$

is a local maximum. The expected utility is

$$\frac{p_2^2 \pi_1^{\frac{2}{3}}}{4 p_1 \pi_2^{\frac{2}{3}}}.$$

Finally, comparing the local maxima and $(0, 0)$ show that the allocation $(\hat{x}_1, \hat{x}_2) = (\frac{10}{9}, \frac{2}{3})$ is optimal for $p_1 = \frac{2}{3}$ and $\pi_1 = \frac{3}{4}$.

B Proof of Lemma 2

Proof. By way of contradiction, we assume that there is an optimal allocation X with two states s and s' having values in C. We define

$$a := \pi_s X_s + \pi_{s'} X_{s'}$$

and consider the expression

$$f(x) := p_s U(x) + p_{s'} U\left(\frac{a - \pi_s x}{\pi_{s'}}\right).$$

Since U is convex, the same holds true for f. Maximizing a convex function gives a corner point solution. Thus, there is $\tilde{x} \in C$ such that $\frac{a-\pi_s \tilde{x}}{\pi_{s'}} \in C$ holds and $f(\tilde{x}) > f(X_s)$ is satisfied. Let us define a new candidate $\tilde{X}$ by

$$\begin{aligned}
\tilde{X}_s &= \tilde{x}, \\
\tilde{X}_{s'} &= \frac{a - \pi_s \tilde{x}}{\pi_{s'}}, \\
\tilde{X}_{s''} &= X_{s''} \quad \text{for } s'' \in \{1, \dots, S\} \text{ and } s \neq s', s \neq s''.
\end{aligned}$$

By construction, $\tilde{X}$ is still affordable with initial endowment and gives a higher utility. This gives a contradiction to the optimality of X. □

C Example 3

In order to show that $\hat{q} = (1,1)$, $\hat{X}^1 = (\frac{14}{9}, \frac{7}{2}, 14)$ and $\hat{X}^2 = (28, \frac{7}{2}, \frac{28}{19})$ is a financial market equilibrium, we have to check feasibility of the allocation $(\hat{X}^i)_{i\in\mathcal{I}}$ and optimality of $\hat{X}^i$ for the utility maximization problem of agent i. We first solve the utility maximization problem of the agents for prices $\hat{q} = (1,1)$ and we then check that the optimal allocation $(\hat{X}^i)_{i\in\mathcal{I}}$ form a feasible allocation.
In order to maximize the expected utility, agent i chooses a strategy $\theta^i = \left(\theta_0^i, \theta_1^i\right)^t$, i.e., agent i buys θ_j^i of asset j subject to his initial endowment. Formally, this can be described by the optimization problem

$$\begin{aligned}
&\text{maximize } \sum_{k=1}^{3} p_k \ln\left(W_k^i + \theta_0^i + \theta_1^i A_{1k}\right) \\
&\text{subject to } \theta_0^i + \theta_1^i q_1 \leq 0 \text{ and } W^i + \theta_0^i + \theta_1^i A_1 > 0
\end{aligned}$$

for agent i. In the optimization problems, the initial endowment W^i, the price q_1 and the probabilities are fixed. Due to the monotonicity of logarithm, we can replace the inequality in $\theta_0^i + \theta_1^i q_1 \leq 0$ by equality and replace θ_0^i by $-\theta_1^i \cdot q_1$. This simplifies the optimization problem to a maximization of a function depending on θ_1^i. The boundary condition $W^i + \theta_0^i + \theta_1^i A_1 = W^i + \theta_1^i(A_1 - q_1) > 0$ has to be satisfied for every state s. The prices $\hat{q} = (1, 1)$ exclude arbitrage and it follows that $A_1 - q_1$ is both positive and negative for at least one state. We conclude that the boundary condition define a bounded interval of possible values for θ_1^i. The property $\ln(0) = -\infty$ implies that a candidate that satisfies $W_k^i + \theta_0^i + \theta_1^i A_{1k} = 0$ in at least one coordinate, can not be optimal. Hence, a solution exists and satisfies the first-order conditions. Differentiating the function $\sum_{k=1}^3 p_k \ln\left(W_k^i + \theta_1^i(A_{1k} - q_1)\right)$ with respect to θ_1^i and setting the resulting term equal to 0 give

$$\frac{A_{11} - q_1}{W_1^1 - q_1\theta_1^1 + \theta_1^1 A_{11}} + \frac{A_{12} - q_1}{W_2^1 - q_1\theta_1^1 + \theta_1^1 A_{12}} + \frac{A_{13} - q_1}{W_3^1 - q_1\theta_1^1 + \theta_1^1 A_{13}} = 0 \quad (3)$$

for agent 1 and

$$\frac{A_{11} - q_1}{W_1^2 - q_1\theta_1^2 + \theta_1^2 A_{11}} + \frac{A_{12} - q_1}{W_2^2 - q_1\theta_1^2 + \theta_1^2 A_{12}} + \frac{A_{13} - q_1}{W_3^2 - q_1\theta_1^2 + \theta_1^2 A_{13}} = 0 \quad (4)$$

for agent 2. Plugging in the explicit numbers for the price, the payoffs and the initial endowments and solving the equations for θ_1^i show that 1 and $\frac{649}{19}$ solve equation (3) and $\frac{2648}{19}$ and -1 solve equation (4). However, $\theta_1^1 = \frac{649}{19}$ violates the boundary condition $W^1 + \theta_1^1(A_1 - q_1) > 0$ in state 1 and $\theta_1^2 = \frac{2648}{19}$ violates the boundary condition $W^2 + \theta_1^2(A_1 - q_1) > 0$ in state 3. We conclude that $\theta_1^1 = 1$ and $\theta_1^2 = -1$ solve the utility maximization problems of the agents. In particular, we see that $\sum_{i \in \mathcal{I}} \theta_1^i = 0$ holds, i.e., the market clearing condition is also satisfied.
In order to show uniqueness of the equilibrium, we solve equation (3) and (4) for a general price q_1. This gives again multiple solutions θ_1^{1+} and θ_1^{1-} for (3) and θ_1^{2+} and θ_1^{2-} for (4). Thus, there are four possible combinations and every combination determines an equilibrium price q_1 via the market-clearing condition $\sum_{i \in \mathcal{I}} \theta_1^i = 0$:

- *Case* $+/+$: The $+/+$-combination gives the price $q_1 = 1$, which we already analyzed above.

- *Case* $-/+$: The market-clearing condition gives the price $q_1 \approx 1.2461$. It follows that $\theta_1^1 \approx 23.07$, and $\theta_0^1 = -q_1\theta_1^1 \approx -28.7527$. This implies $W^1 + \theta_0^1 + \theta_1^1 A_1 < 0$ in state 1, i.e., the boundary condition is violated. Hence, it cannot be an equilibrium.

- *Case* $+/-$: The market-clearing condition gives the price $q_1 \approx 0.3412$. It follows that $\theta_1^1 \approx -18.99$, $\theta_1^2 = -\theta_1^1$ and $\theta_0^1 = -q_1\theta_1^1 \approx 6.4834$. This implies $W^1 + \theta_0^1 + \theta_0^1 A_1$ in state 1, i.e., the boundary condition is violated. Hence, it cannot be an equilibrium.

- *Case* $-/-$: The market-clearing condition has no solution.

We conclude that only the $+/+$-combination leads to an equilibrium, which is the one we already analyzed above.

D Example 4

In order to derive the explicit form of the price q_1, we follow a similar path as in Appendix C. In order to maximize the expected utility, agent i chooses a strategy $\theta^i = \left(\theta_0^i, \theta_1^i\right)^t$, i.e., agent i buys θ_j^i of asset j subject to his initial endowment. Formally, this can be described by the optimization problems

$$\begin{aligned}&\text{maximize } p_1^1 \ln\left(W_1^1 + \theta_0^1\right) + p_2^1 \ln\left(W_2^1 + \theta_1^1\right)\\ &\text{subject to } \theta_0^1 + \theta_1^1 q_1 \leq 0 \text{ and } W_k^1 + \theta_0^1 A_{0k} + \theta_1^1 A_{1k} > 0 \text{ for } k = 1, 2\end{aligned}$$

for agent 1 and

$$\begin{aligned}&\text{maximize } p_1^2 \ln\left(W_1^2 + \theta_0^2\right) + p_3^2 \ln\left(W_3^2 + \theta_1^2\right)\\ &\text{subject to } \theta_0^2 + \theta_1^2 q_1 \leq 0 \text{ and } W_k^2 + \theta_0^2 A_{0k} + \theta_1^2 A_{1k} > 0 \text{ for } k = 1, 3\end{aligned}$$

for agent 2. Due to the monotonicity of ln, we can replace the inequality in $\theta_0^i + \theta_1^i q_1 \leq 0$ by equality and replace θ_0^i by $-\theta_1^i \cdot q_1$. This simplifies the optimization problem to a maximization of a function depending on θ_1^i. Note that the (state-wise) boundary conditions $W^i + \theta_0^i + \theta_1^i A_1 > 0$ give a bounded interval of possible values for θ_1^i. Because of $\ln(0) = -\infty$, a candidate that satisfies $W_k^i + \theta_0^i + \theta_1^i A_{1k} = 0$ in a state with strictly positive subjective probability can not be optimal. Thus, we can reduce to allocations satisfying the first-order condition. Differentiating the function with respect to θ_1^i and setting the resulting term equal to 0 give

$$\frac{-q_1}{W_1^1 - \theta_1^1 q_1} + \frac{1}{W_2^1 + \theta_1^1} = 0$$

for agent 1 and

$$\frac{-q_1}{W_1^2 - \theta_1^2 q_1} + \frac{1}{W_3^2 + \theta_1^2} = 0$$

for agent 2. These equations can be solved for θ_1^1 and θ_1^2 and we end up with

$$\begin{aligned}\theta_1^1 &= \frac{W_1^1 - q_1 W_2^1}{2q_1},\\ \theta_1^2 &= \frac{W_1^2 - q_1 W_3^2}{2q_1}.\end{aligned}$$

In order to "clear away" any excess supply and excess demand, the quantity demanded and the quantity supplied should be equal. In our setup, this means that $\theta_0^1 = -\theta_0^2$ and $\theta_1^1 = -\theta_1^2$ have to hold. We deduce the explicit form

$$q_1 = \frac{W_1^1 + W_1^2}{W_3^2 + W_2^1}$$

of the price of the risky asset.

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